2. [12 points] Consider a particle whose trajectory in the xy-plane is given by the parametric curve defined by the equations

$$x(t) = t^4 - 4t^2, \qquad y(t) = t^2 - 2t,$$

for $-3 \le t \le 3$. Show all your work to receive full credit.

a. [3 points] Is there any value of t at which the particle ever comes to a stop? Justify.

Solution: No. For the particle to come a stop, its velocity in both the x- and y-direction must be zero. We have that

$$\frac{dx}{dt} = 4t^3 - 8t = 4t(t^2 - 2) = 0$$

at $t = 0, \pm\sqrt{2}$ and

$$\frac{dy}{dt} = 2t - 2 = 0$$

at $t = 1$. Since there are no times at which $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously zero, the particle never comes to a stop.

b. [2 points] For what values of t does the path of the particle have a vertical tangent line?

Solution: Vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. From the above calculation, this is true at $t = 0, \pm\sqrt{2}$.

c. [3 points] What is the lowest point (x, y) on the curve?

Solution: We want to minimize the value of the y-coordinate over $-3 \le t \le 3$. The only critical point for $y(t)$ was found above at $t = 1$. Since $\left.\frac{dy}{dt}\right|_{t=0} = -2 < 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = 2 > 0$, the First Derivative Test tells us that $t = 1$ is a local minimum, and thus a global minimum since it is the only critical point on the given interval. The lowest point on the curve is thus $(x(1), y(1)) = (-3, -1)$.

d. [2 points] At what values of t does the particle pass through the origin?

Solution: We set $x(t) = 0$ and $y(t) = 0$ and solve for t.

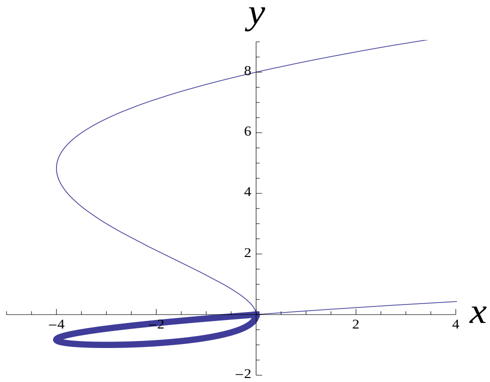
$$x(t) = t^4 - 4t^2 = t^2(t^2 - 4) = t^2(t - 2)(t + 2) = 0$$

gives that $t = -2, 0, 2$, while

$$y(t) = t^2 - 2t = t(t - 2) = 0$$

gives $t = 0, 2$.
Thus, the particle passes through the origin at times $t = 0$ and $t = 2$.

e. [2 points] The graph of the curve traced by these parametric equations is shown below. Find an expression for the length of the closed loop marked in the graph.



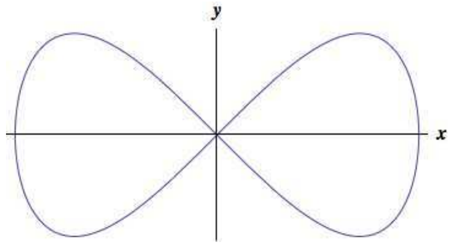
Solution: From the given graph and above calculation, we know that the loop is traced out over the time interval $0 \le t \le 2$. The arclength of the loop is given by

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(4t^3 - 8t)^2 + (2t-2)^2}\, dt.$$

6. [13 points] A particle moves along the path given by the parametric equations

$$x(t) = a\cos t \qquad y(t) = \sin 2t \qquad \text{for } 0 \le t \le 2\pi.$$

where a is a positive constant. The graph of the particle's path in the x-y plane is shown below. In the questions below, show all your work to receive full credit.



a. [2 points] At which values of $0 \le t \le 2\pi$, does the particle pass through the origin?

Solution: $0 = x(t) = a\cos t$: $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
$0 = y(t) = \sin 2t$: $2t = 0, \pi, 2\pi, 3\pi, 4\pi \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. Particle passes through origin: $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

b. [5 points] For what values of a are the two tangent lines to the curve at the origin perpendicular? Hint: Two lines are perpendicular if the product of their slopes is equal to -1.

Solution: $x'(t) = -a\sin t$, $y'(t) = 2\cos 2t$.

$t = \frac{\pi}{2}$	$t = \frac{3\pi}{2}$
$\frac{dx}{dt} = -a$	$\frac{dx}{dt} = a$
$\frac{dy}{dt} = -2$	$\frac{dy}{dt} = -2$
$\frac{dy}{dx} = \frac{2}{a}$	$\frac{dy}{dx} = -\frac{2}{a}$

$$-1 = \frac{2}{a}\left(-\frac{2}{a}\right) = -\frac{4}{a^2} \Rightarrow a = 2.$$

c. [4 points] At what values of $0 \le t \le 2\pi$, does the curve have horizontal tangents?

Solution:

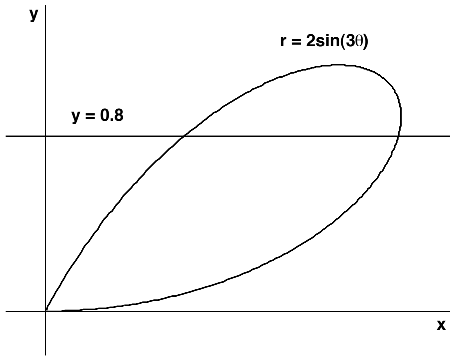
$$0 = y'(t) = 2\cos 2t \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots, \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

d. [2 points] Find an expression that computes the length of the curve.

Solution:

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + 4\cos^2 2t}\,dt.$$

8. [14 points] Consider the area contained above the line $y = 0.8$, and below the curve $r = 2\sin(3\theta)$. You may find the following figure helpful.



a. [4 points] Find the (x, y) coordinates for the two points where $y = 0.8$ and $r = 2\sin(3\theta)$ intersect as shown in the figure above. Show enough work to support your answer.

Solution: We convert $y = 0.8$ into $r = \frac{0.8}{\sin\theta}$, then solve for $\frac{0.8}{\sin\theta} = 2\sin(3\theta)$, which gives $\theta \approx 0.4296, 0.8623$. The corresponding coordinates are $(1.7464, 0.8)$ and $(0.6854, 0.8)$.

b. [4 points] Write an expression for the area that is specified. You do not need to evaluate your expression.

Solution:

$$\text{Area} = \frac{1}{2}\int_{.4296}^{.8623} 4\sin^2(3\theta)d\theta - \frac{1}{2}\int_{.4296}^{.8623} \frac{0.64}{\sin^2(\theta)}d\theta$$

c. [6 points] Calculate the perimeter that surrounds the specified area. You may round your final answer to two decimal places.

Solution: The distance along the line $y = 0.8$, is $1.7464 - 0.6854 = 1.0610$. Along the curve $r = 2\sin(3\theta)$, $x = 2\sin(3\theta)\cos(\theta)$ and $y = 2\sin(3\theta)\sin(\theta)$. We use this to find

$$\frac{dx}{d\theta} = -2\sin(3\theta)\sin(\theta) + 6\cos(3\theta)\cos(\theta) \text{ and } \frac{dy}{d\theta} = 2\sin(3\theta)\cos(\theta) + 6\cos(3\theta)\sin(\theta).$$

$$\text{arc length} = \int_{.4296}^{.8623} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \approx 1.3737$$

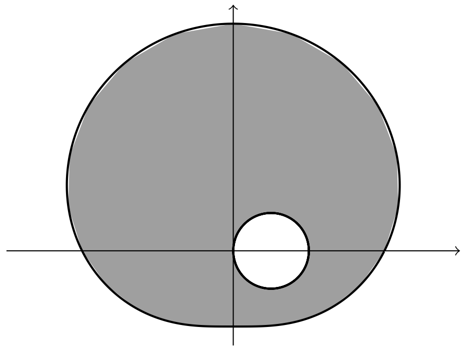
The total perimeter is approximately $1.3737 + 1.0610 = 2.4347 \approx 2.43$.

5. [11 points] Franklin's robot army is surrounding you!

a. [6 points] Consider the polar curves

$$r = \cos(\theta) \qquad\qquad r = \sin(\theta) + 2$$

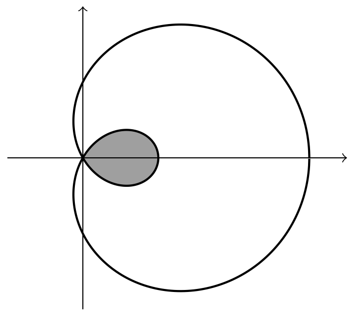
Franklin's robot army occupies the shaded region between these two curves. Write an expression involving integrals that gives the **area** occupied by Franklin's robot army. Do not evaluate any integrals.



Solution:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\sin(\theta) + 2)^2 d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 d\theta$$

b. [5 points] Your friend, Kazilla, pours her magic potion on the ground. Suddenly, a flock of wild chickens surrounds you. The chickens occupy the shaded region enclosed within the polar curve $r = 1 + 2\cos(\theta)$ as shown below. Write an expression involving integrals that gives the **perimeter** of the region occupied by the flock of wild chickens. Do not evaluate any integrals.



Solution: We use the arc length formula:

$$\text{Arc Length} = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2} d\theta$$

Note that $r'(\theta) = -2\sin(\theta)$. Also, the shaded region of lies between $\theta = 2\pi/3$ and $\theta = 4\pi/3$ (you can see this by setting $r(\theta) = 0$, and testing that $r(\pi) = -1$, so it lies on the boundary of the shaded region.)

$$\text{Arc Length} = \int_{2\pi/3}^{4\pi/3} \sqrt{(1 + 2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta$$