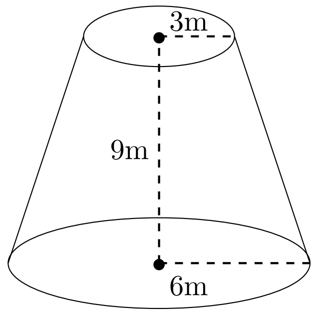
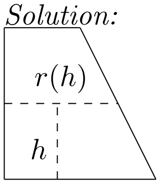
9. [12 points] The Nub's Nob Ski Area keeps a massive supply of hot chocolate. The hot chocolate is stored in a container shaped like a cone with the point end removed as shown below. The height of the container is 9 meters, and it has lower radius 6 meters and upper radius 3 meters. The hot chocolate has a density of 3000 kg/m^3. Recall the gravitational constant is $g = 9.8\text{m/s}^2$.



a. [3 points] Write a formula for $r(h)$, the radius of a circular cross section of the container h meters above the base.

Solution:



Looking at a vertical cross section of the cone we see that $r(h)$ is the width of a trapezoid at height h. The width of the trapezoid is decreasingly linearly thus $r(h)$ must be a linear function with $r(0) = 6$ and $r(9) = 3$. Therefore $r(h) = 3 + \frac{3(9-h)}{9} = 6 - h/3$.

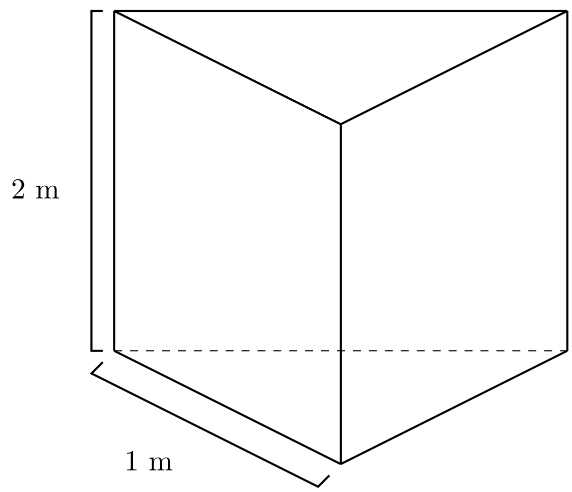
b. [6 points] Write a formula in terms of $r(h)$ for the work required to lift a slice of hot chocolate of thickness Δh from height h to the top of the container.

Solution: The mass of the slice is $3000\pi r(h)^2\Delta h$. The slice must be lifted $9 - h$ meters. Therefore the work to lift the slice is $3000g\pi r(h)^2(9 - h)\Delta h$.

c. [3 points] Write an integral that gives the work required to lift all of the hot chocolate to the top of the container. Do not evaluate this integral.

Solution: Integrating the above function from 0 to 9 the work is $\int_0^9 3000g\pi r(h)^2(9-h)dh$

9. [9 points] The tank pictured below has height 2 meters, and the top and bottom are equilateral triangles with sides of length 1 meter. It is filled **halfway** with hot chocolate. The hot chocolate has uniform density 1325 kg/m^3. The acceleration due to gravity is 9.8 m/s^2. Calculate the work needed to pump all the chocolate to the top of the tank. Show all your work. Give an **exact** answer. Include **units**.



Solution: We take a horizontal slice at height y meters from the bottom of the tank. It has mass $1325 \cdot \frac{\sqrt{3}}{4} 1^2 \Delta y$. We need to move it $2 - y$ meters up. Thus, the work needed to pump all the chocolate to the top is

$$\int_0^1 1325 \cdot \frac{\sqrt{3}}{4} 1^2 \cdot 9.8 \cdot (2-y)\, dy = \frac{1325 \cdot 9.8\sqrt{3}}{4}\left[2y - \frac{y^2}{2}\right]_{y=0}^{y=1} = \frac{1325 \cdot 9.8\sqrt{3}}{4} \cdot \frac{3}{2} \text{ Joules}$$