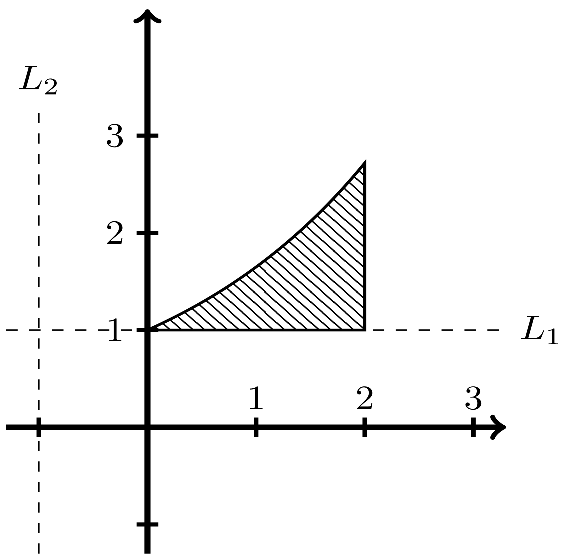
**6**. [12 points]
The region bounded by the graph of $y = e^{0.5x}$, the line $y = 1$, and the line $x = 2$ is shown below. For each of the lines $L_1$ and $L_2$ write a definite integral that represents the volume of the solid object obtained by rotating the region around that line. You do not need to show your work or calculate the value of the integral.




**a**. [6 points] $L_1$:

*Solution:*

•Disks:

$$\int_0^2 \pi(e^{0.5x} - 1)^2 dx$$

•Washers: Intersection point $(2, e)$

$$\int_1^e 2\pi(y-1)(2 - 2\ln y)dy$$

**b**. [6 points] $L_2$:

*Solution:*

•Disks: Intersection point $(2, e)$

$$\int_1^e \pi((2+1)^2 - (2\ln(y) + 1)^2)dy$$

•Washers:

$$\int_0^2 2\pi(1+x)(e^{0.5x} - 1)dx$$

**9**. [10 points] Jennifer is designing a doorknob. The shape of the doorknob is the solid formed by rotating the region bounded by $y = 2 - \cos(2x)$, $y = \frac{1}{4}$, $x = \frac{\pi}{2}$, and the $y$-axis about the $x$-axis. Assume the units of $x$ and $y$ are inches.

**a**. [5 points] Write an integral which gives the volume of the doorknob. Do not evaluate your integral. Circle your answer.

*Solution:* $\int_0^{\pi/2} \pi[(2 - \cos(2x))^2 - (1/4)^2]\, dx$

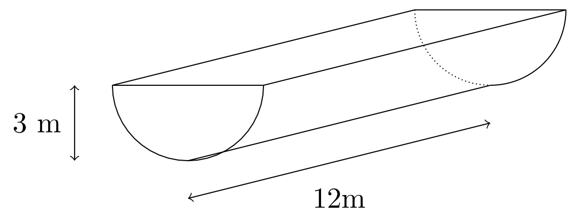
**b**. [5 points] The doorknob is to be made out of a material with constant density $\delta$. The $y$-coordinate of the center of mass of the doorknob is $\bar{y} = 0$. Write an expression involving integrals which gives the $x$-coordinate of the center of mass of the doorknob. Do not evaluate your expression. Circle your answer.

*Solution:* $\frac{\int_0^{\pi/2} \pi x[(2-\cos(2x))^2-(1/4)^2]\, dx}{\int_0^{\pi/2} \pi[(2-\cos(2x))^2-(1/4)^2]\, dx}$

**5**. [11 points] Robber baron and philanthropist Calvin Currency is making a large cash donation in \$100 bills. Before making the donation, he decides to fill an empty pool with the money. The pool is a half cylinder with radius 3 meters and length 12 meters as shown below. After an afternoon of diving into his pool of money and swimming around, the distribution of bills in the pool becomes nonuniform and so the density of money in the pool is given by

$$\delta(y) = 30,000\sqrt{\frac{10}{\pi}}e^{-y^2},$$

measured in bills per $m^3$, where $y$ is height in meters measured from the bottom of the pool. Recall the gravitational constant is $g = 9.8$ m/s$^2$




**a**. [5 points] Write a definite integral which gives the volume of the pool.

*Solution:* The volume of the pool is $\int_0^3 12(2\sqrt{9-(y-3)^2})dy$.

**b**. [2 points] Write a definite integral which gives the value of the money in the pool, in dollars.

*Solution:* The value of money in the pool is given by $100\int_0^3 \delta(y)(12)(2\sqrt{9-(y-3)^2})dy$.

**c**. [4 points] Write a definite integral which gives the amount of work done in lifting the money out of the pool if each bill has mass 0.001 kg.

*Solution:* The work done in lifting the money out of the pool is given by
$\int_0^3 (0.001)\delta(y)(g)(12)(2\sqrt{9-(y-3)^2})(3-y)dy$.

**4**. [16 points] Consider the region $R$ bounded by the graphs of $y = \ln(x)$, $y = 0$ and $x = 2$. In the following questions, show all your work to receive full credit.

**a**. [4 points] Find the perimeter of the region $R$. You may use your calculator to evaluate any integrals.

*Solution:* $L = 1 + \ln 2 + \int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx \approx 2.915.$

**b**. [5 points] Let $S$ be the solid obtained by rotating the region $R$ about the $y$ axis. Write an expression for the volume of a slice of the solid $S$ located at a height $y$ with thickness $\Delta y$.

*Solution:* $V_{slice} \approx \pi[4 - e^{2y}]\Delta y.$

**c**. [2 points] Suppose $S$ has mass density $\delta(y) = e^{-y}$. Write an expression for the mass of the solid $S$ using a definite integral. You do not need to evaluate this integral.

*Solution:* $\text{Mass} = \int_0^{\ln 2} e^{-y}\pi[4 - e^{2y}]dy.$

**d**. [2 points] What is the value of $\bar{x}$, the $x$ coordinate of the center of mass of $S$? Justify.

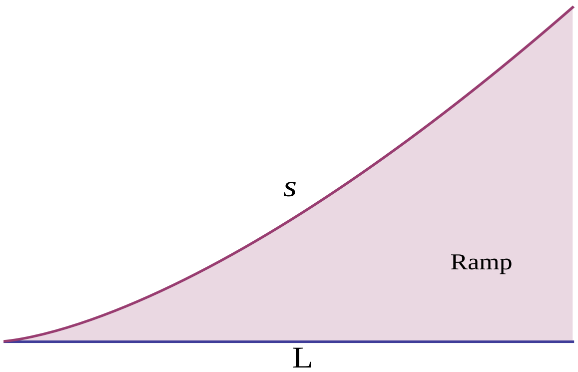
*Solution:* $S$ is a solid of revolution where the $y$ axis is its axis of symmetry and the mass density $\delta(y)$ is independent of $x$, hence the center of mass should be on the $y$ axis. Hence $\bar{x} = 0$.

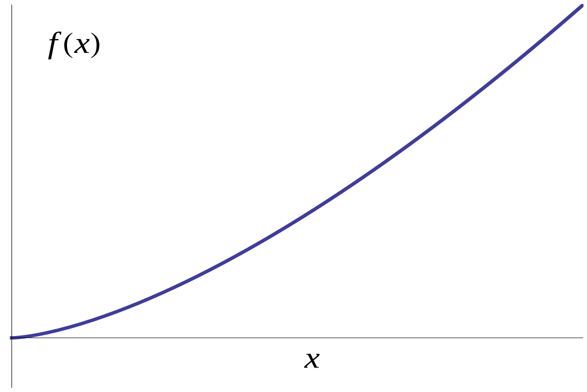
**e**. [3 points] Write an expression for $\bar{y}$, the $y$ coordinate of the center of mass of $S$, using definite integrals. You do not need to evaluate this expression.

*Solution:*

$$\bar{y} = \frac{\int_0^{\ln 2} y e^{-y}\pi[4 - e^{2y}]dy}{\int_0^{\ln 2} e^{-y}\pi[4 - e^{2y}]dy}.$$

**5**. [8 points] A company wants to design a bicycle ramp using the shape of the graph of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}}$,where $x$ is the length in meters of the base of the ramp.



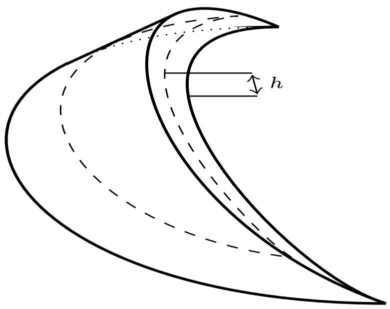



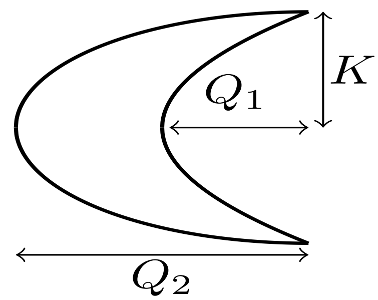

Find the length $s$ of a ramp with base of length $L$. Show all your work.

*Solution:*

$$\begin{aligned} s &= \int_0^L \sqrt{1+(f'(x))^2}dx \\ &= \int_0^L \sqrt{1+(2\sqrt{x})^2}dx \\ &= \int_0^L \sqrt{1+4x}dx \\ &= \frac{1}{6}(1+4x)^{3/2}|_0^L \\ &= \frac{1}{6}(1+4L)^{3/2} - \frac{1}{6} \end{aligned}$$

**8.** [12 points] Sand dunes come in many shapes. *Barchan* dunes, which have the shape shown on the left, are studied extensively by geomorphologists. Horizontal cross-sections of these dunes are crescent-shaped (the dashed line encloses one such cross-section), and can be approximated as the shape on the right. The area of this shape is given by the formula $A_h = K(\frac{\pi}{2}Q_2 - \frac{4}{3}Q_1)$.




You are studying a barchan dune of 10 meters height, for which the values of $Q_1$, $Q_2$, and $K$ vary with respect to the height $h$ (in meters) of the cross-section according to the functions $Q_1(h) = 10 - h$, $Q_2(h) = 20 - 2h$, $K(h) = 100 - h^2$. The density of sand in the dune is $\delta = 1600$ kilograms per cubic meter.

**a.** [5 points] Write an expression for the volume of one slice of sand dune $h$ meters above the ground and $\Delta h$ meters thick.

*Solution:*

$$V_{slice} \approx A_h \Delta h = (100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right]\Delta h.$$

**b.** [5 points] Write a definite integral that represents the total mass of sand in the dune. You do not need to evaluate this integral.

*Solution:* Height of the dune = 10, so

$$M_{dune} = \int_0^{10} 1600(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh.$$

**c.** [2 points] Write an expression (involving integrals) for the height of the center of mass of the sand dune. You do not need to evaluate this integral.

*Solution:*

$$\bar{h} = \frac{\int_0^{10} 1600h(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}{\int_0^{10}(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}$$