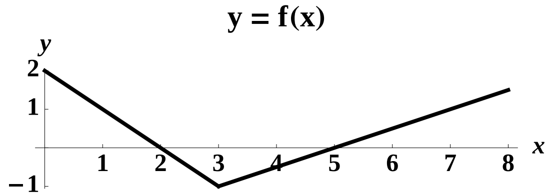
5. [12 points] The graph of $f(x)$ and a table of values for the continuous functions $g(x)$ and $h(x)$ are given below. The function $h(x)$ is an antiderivative of $g(x)$.



x	0	1	2	3	4
$g(x)$	1	3	5	7	9
$h(x)$	-3	-1	3	9	17

Compute the **exact** value of each of the following expressions:

a. [1 point] $\int_0^7 |f(x)|dx$

Solution: $\int_0^7 |f(x)|dx = 2 + \frac{3}{2} + 1 = \frac{9}{2}$.

b. [4 points] $\int_1^{e^2} \frac{f(\ln x)}{x} dx$

Solution: If $u = \ln x$, then

$$\int_1^{e^2} \frac{f(\ln x)}{x} dx = \int_0^2 f(u)du = 2$$

c. [7 points] Find $\int_1^2 xg'(2x)dx$

Solution: If $w = 2x$, then

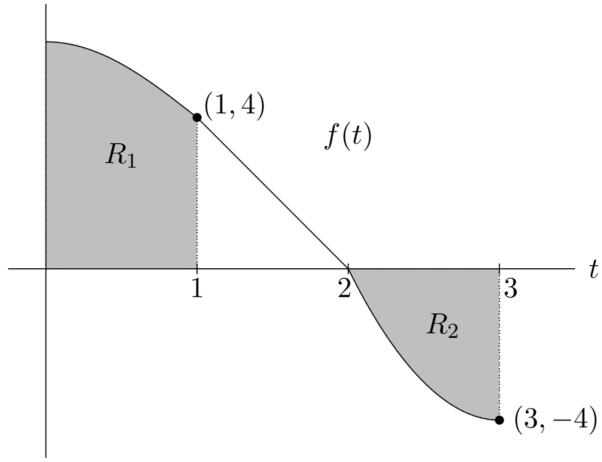
$$\int_1^2 xg'(2x)dx = \frac{1}{2}\int_2^4 \frac{w}{2} g'(w)dw = \frac{1}{4}\int_2^4 wg'(w)dw.$$

Integration by parts with $u = w$ and $v' = g'(w)$ yields

$$\begin{aligned}\int_2^4 wg'(w)dw &= wg(w)\,\Big|_2^4 - \int_2^4 g(w)dw \\ &= 4g(4) - 2g(2) - (h(4) - h(2)) \\ &= 4(9) - 2(5) - (17 - 3) = 12.\end{aligned}$$

$$\int_1^2 xg'(2x)dx = \frac{1}{4}\int_2^4 wg'(w)dw = \frac{1}{4}(12) = 3.$$

1. [14 points] While you are trying to fill your old bucket with water, it begins to leak. Suppose the continuous function $f(t)$ is the rate of change of the volume of water in the bucket, in gallons per minute, t minutes after it begins to leak. A graph of $f(t)$ for $0 \le t \le 3$ is shown below. The function $f(t)$ is linear for $1 \le t \le 2$. The region R_1 has area 5.8, and the region R_2 has area 3. There are 7 gallons of water in the bucket at $t = 1$.



a. [5 points] Write an expression involving integrals for $A(t)$, the volume of water in the bucket, in gallons, t minutes after the bucket began to leak where $0 \le t \le 3$. Your expression may contain the function f.

Solution: $A(t) = 7 + \int_1^t f(x)dx$

b. [2 points] How much water was in the bucket when it began to leak? How much water was in the bucket 3 minutes after it began to leak? Fill in the blanks below.

Solution:

There were <u>1.2</u> gallons of water in the bucket when it began to leak.

There were <u>6</u> gallons of water in the bucket 3 minutes after it began to leak.

c. [3 points] Write an expression involving an integral for the average rate of change of the amount of water in the bucket during the first three minutes after it began to leak, and find the value of your expression, including units.

Solution: $\frac{1}{3-0}\int_0^3 f(t)dt = \frac{1}{3}[A(3) - A(0)] = \frac{4.8}{3}$ gal/min.

d. [4 points] For $t \geq 3$, suppose $f(t)$ is linear with slope 1, but is only defined until the time when the bucket is empty. For what value of t is the bucket empty? (Remember that f is continuous as specified above).

Solution: For $t \geq 3$, $f(t)$ is linear with slope 1 and passes through the point $(3, -4)$. So, $f(t) = t - 7$ for $t \geq 3$ until the time when the bucket is empty. To find the t value where the bucket is empty, we can solve $\displaystyle\int_3^t x - 7dx = -6$ to get $t^2 - 14t + 45 = 0$. We can factor the quadratic polynomial in t to get that $t = 5$ or $t = 9$. Then $f(t)$ will be defined until $t = 5$.

5. [10 points] For each statement below, circle TRUE if the statement is *always* true; otherwise, circle FALSE. There is no partial credit on this page.

a. [2 points] The function $\frac{\sin x}{x}$ has an anti-derivative.

[True] False

b. [2 points] $\frac{d}{dx}\int_x^{x^2} e^{t^2}\,dt = 4x^3e^{x^4} - 2xe^{x^2}$.

True [False]

c. [2 points] The average of the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 3$ is $\ln(\sqrt{3})$.

[True] False

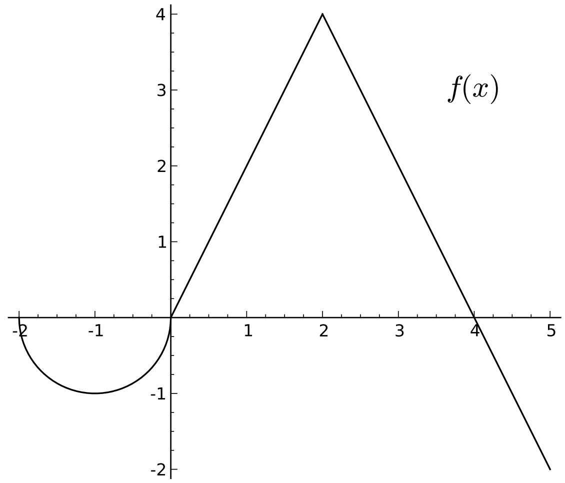
d. [2 points] $\int_a^b f(x)dx$ is greater than or equal to at least one of LEFT(n), RIGHT(n), TRAP(n), or MID(n) regardless of what $f(x)$ or n is.

True [False]

e. [2 points] If $\int_a^b f(x)dx > 0$ then $f(b) > f(a)$.

True [False]

2. [18 points] The graph of the function $f(x)$, shown below, consists of line segments and a semicircle. Compute each of the following quantities.



a. [7 points]

1. $\int_0^2 f(x)dx = 4.$

2. $\int_{-2}^2 |f(x)|dx = \frac{\pi}{2} + 4 \approx 5.57.$

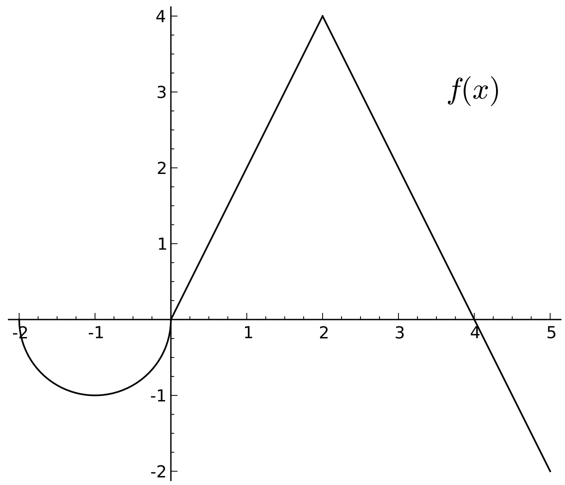
3. $\int_0^5 f(x)dx = 8 - 1 = 7.$

4. $\int_{-2}^2 2f(x)dx + \int_5^2 3f(x)dx = 2(4 - \frac{\pi}{2}) - 3(4-1) = -1 - \pi \approx -4.14.$

5. The average A of $f(x)$ on the interval $[-2, 5]$. $A = \frac{1}{7}\int_{-2}^5 f(x)dx = \frac{7 - \frac{\pi}{2}}{7} \approx .775.$

6. $\int_0^1 f(5x)dx = \frac{1}{5}\int_0^5 f(u)du = \frac{7}{5}.$

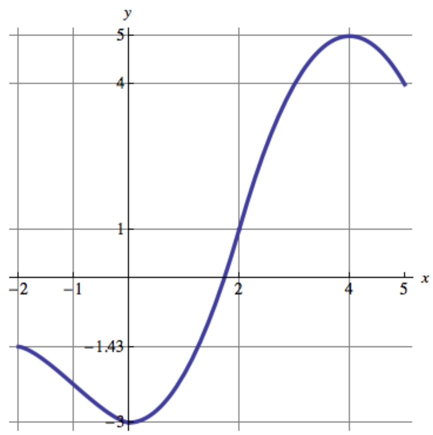
b. [4 points]



If $f(x)$ is the derivative of a function $g(x)$ with $g(2) = 1$, fill in the table of values of $g(x)$, provided below, at the specified points (the graph has been reproduced for your convenience):

x	-2	0	2	4	5
$g(x)$	$\frac{\pi}{2} - 3 \approx$ -1.429	-3	1	5	4

c. [5 points] Graph $g(x)$. Make sure your graph indicates the intervals on which $g(x)$ is increasing, decreasing, concave up, and concave down.



d. [2 points] Let $h(x) = \int_0^x f(t)dt$. Find a constant C such that $g(x) = h(x) + C$. Show all your work.

$$g(x) = 1 + \int_2^x f(t)dt = 1 + \int_0^x f(t)dt - \int_0^2 f(t)dt = 1 + h(x) - 4$$
$$g(x) = h(x) - 3.$$

$C = -3$.

3. [13 points] Use the table and the fact that

$$\int_0^{10} f(t)dt = 350$$

to evaluate the definite integrals below exactly (i.e., no decimal approximations). Assume $f'(t)$ is continuous and does not change sign between any consecutive t-values in the table.

t	0	10	20	30	40	50	60
$f(t)$	0	70	e^5	e^3	0	$\pi/2$	π

a. [4 points] $\int_0^{10} tf'(t)dt$

Solution:

$$\begin{aligned}\int_0^{10} tf'(t)dt &= tf(t)|_0^{10} - \int_0^{10} f(t)dt\\ &= 10f(10) - \int_0^{10} f(t)dt\\ &= 700 - 350\\ &= 350.\end{aligned}$$

b. [4 points] $\int_{20}^{30} \frac{f'(t)}{f(t)}dt$

Solution:

$$\begin{aligned}\int_{20}^{30} \frac{f'(t)}{f(t)}dt &= \int_{f(20)}^{f(30)} \frac{1}{u}du\\ &= \ln|u||_{f(20)}^{f(30)}\\ &= \ln|f(30)| - \ln|f(20)|\\ &= 3 - 5\\ &= -2.\end{aligned}$$

c. [5 points] $\int_{50}^{60} f(t)f'(t)\sin(f(t))dt$

Solution:

$$\begin{aligned}\int_{50}^{60} f(t)f'(t)\sin(f(t))dt &= \int_{f(50)}^{f(60)} w\sin(w)dw\\ &= -w\cos(w)|_{f(50)}^{f(60)} + \int_{f(50)}^{f(60)} \cos(w)dw\\ &= -\pi\cos(\pi) + \int_{\pi/2}^{\pi} \cos(w)dw\\ &= \pi - 1\end{aligned}$$

1. [16 points] At a time t seconds after a catapult throws a rock, the rock has horizontal velocity $v(t)$ m/s. Assume $v(t)$ is monotonic between the values given in the table and does not change concavity.

t	0	1	2	3	4	5	6	7	8
$v(t)$	47	34	24	16	10	6	3	1	0

a. [4 points] Estimate the average horizontal velocity of the rock between $t = 2$ and $t = 5$ using the trapezoid rule with 3 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\frac{\int_2^5 v(t)\,dt}{5-2} = \frac{Left(3) + Right(3)}{2 \cdot 3} = \frac{(v(2)+v(3)+v(4)) + (v(3)+v(4)+v(5))}{6} =$$

$$= \frac{24+16+10+16+10+6}{6} = \frac{82}{6} = \frac{41}{3}$$

The average horizontal velocity of the rock is 41/3 m/s.

b. [4 points] Estimate the total horizontal distance the rock traveled using a left Riemann sum with 8 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Left(8) = v(0)+v(1)+v(2)+v(3)+v(4)+v(5)+v(6)+v(7) =$$

$$= 47+34+24+16+10+6+3+1 = 141$$

The total horizontal distance the rock traveled is approximately 141 meters.

c. [4 points] Estimate the total horizontal distance the rock traveled using the midpoint rule with 4 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Mid(4) = 2(v(1) + v(3) + v(5) + v(7)) =$$

$$= 2(34 + 16 + 6 + 1) = 114$$

The total horizontal distance the rock traveled is approximately 114 meters.

d. [4 points] A second rock thrown by the catapult traveled horizontally 125 meters. Determine whether the first rock or the second rock traveled farther, or if there is not enough information to decide. Circle your answer. Justify your answer.

Solution:

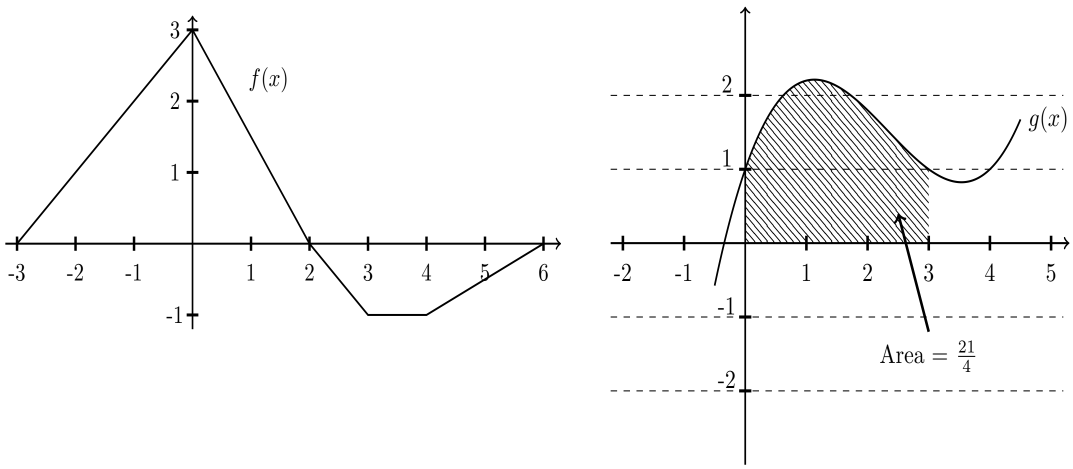
the first rock | **[the second rock]** | not enough information

The function $v(t)$ is concave up since for example

$$-10 = \frac{v(2) - v(1)}{2 - 1} > \frac{v(1) - v(0)}{1 - 0} = -13$$

The trapezoid rule gives $Trap(4) = 121$ (or $Trap(8) = 117.5$). Since $v(t)$ is concave up, this is an overestimate which means that the first rock traveled at most 121 meters, that is less than the second.

3. [15 points] Use the graphs of $f(x)$ and $g(x)$ to find the EXACT values of A,B, and C. Show all your work.



a. [2 points] $A = \int_{-3}^{6} |f(x)|dx$

Solution: $A = \frac{15}{2} + \frac{5}{2} = 10$

b. [5 points] $B = \int_{0}^{2} xg'(x^2)dx$

Solution: Using $u = x^2$
$B = \frac{1}{2}\int_0^4 g'(u)du = \frac{1}{2}\left(g(4) - g(0)\right) = 0$

c. [8 points] $C = \int_0^3 2xg'(x)dx$

Solution: Using integration by parts $C = 2xg(x)|_0^3 - 2\int_0^3 g(x)dx = 6 - 2\left(\frac{21}{4}\right) = -\frac{9}{2}$