5. [14 points] Years later, after being rescued from the island, you design a machine that will automatically feed wood into a fire at a constant rate of 500 pounds per day. At the same time, as it burns, the weight of the wood pile (in pounds) decreases at a rate (in pounds/day) proportional to the current weight with constant of proportionality $\frac{1}{2}$.

a. [3 points] Let $W(t)$ be the weight of the wood pile t days after you start the machine. Write a differential equation satisfied by $W(t)$.

Solution:

$$W' = -0.5W + 500$$

b. [4 points] Find all equilibrium solutions to the differential equation in part (a). For each equilibrium solution, determine whether it is stable or unstable, and give a practical interpretation of its stability in terms of the weight of the wood pile as $t \to \infty$.

Solution: There is one equilibrium solution, $W = 1000$. This equilibrium solution is stable. This means that if your fire initially contains approximately 1000 pounds of wood, then in the long run the weight of the fire will approach 1000 pounds.

c. [7 points] Solve the differential equation from part (a), assuming that the wood pile weighs 200 pounds when you start the machine.

Solution:

$$\begin{aligned}
\frac{dW}{-0.5W + 500} &= dt \\
-2\ln|-0.5W + 500| &= t + C \\
-0.5W + 500 &= Ae^{-0.5t} \\
W &= 1000 + Ae^{-0.5t} \\
W &= 1000 - 800e^{-0.5t}
\end{aligned}$$

7. [13 points] A company designs an air filter for a ship's engine room that reduces the amount of fumes in the air by k percent every hour. The machinery in the engine room produces fumes at a rate of 0.02 kilograms per hour. Let $Q(t)$ be the amount in kilograms of fumes in the room t hours after the engines are activated. Hence Q satisfies

$$\frac{dQ}{dt} = 0.02 - \frac{k}{100}Q.$$

a. [9 points] Find a formula for $Q(t)$. Suppose there are no fumes in the air when the engines are activated.

Solution:

$$\frac{dQ}{0.02 - \frac{k}{100}Q} = dt.$$

$$-\frac{100}{k}\ln\left|0.02 - \frac{k}{100}Q\right| = t + C.$$

$$0.02 - \frac{k}{100}Q = Ae^{-\frac{k}{100}t}$$

$$Q(t) = \frac{100}{k}\left(0.02 - Ae^{-\frac{k}{100}t}\right)$$

$$Q(t) = \frac{2}{k}\left(1 - e^{-\frac{k}{100}t}\right) \qquad \text{using} \qquad Q(0) = 0.$$

b. [2 points] What is the value of $Q(t)$ in the long run?

Solution: $\lim_{t\to\infty} Q(t) = \frac{2}{k}$.

c. [2 points] Air safety regulations require that the *concentration* of fumes in the air not exceed 10^{-4} kilograms per liter at any time. If the volume of air in the engine room is 10^3 liters, for what values of k are the safety regulations met at all times?

Solution: Concentration$=\frac{Q(t)}{\text{Volume}} \leq \frac{\frac{2}{k}}{10^3} \leq 10^{-4}$.

Hence $k \geq 20$.

3. [15 points] A model for cell growth states that the volume $V(t)$ (in mm^3) of a cell at time t (in days) satisfies the differential equation

$$\frac{dV}{dt} = 2e^{-t}V.$$

a. [2 points] Find the equilibrium solutions of this equation.

Solution: $V = 0$.

b. [8 points] Solve the differential equation. The initial volume of the cell is V_0 mm^3. Your answer should contain V_0.

Solution:

$$\begin{aligned}
\frac{dV}{dt} &= 2e^{-t}V \\
\frac{dV}{V} &= 2e^{-t}dt \\
\ln|V| &= -2e^{-t} + C \\
V &= Be^{-2e^{-t}}. \\
V_0 &= Be^{-2} \\
B &= V_0e^2 \\
V &= V_0e^2e^{-2e^{-t}} = V_0e^{2-2e^{-t}}.
\end{aligned}$$

c. [3 points] How long does it take a cell to double its initial size?

Solution:

$$\begin{aligned}
2V_0 &= V_0e^{2-2e^{-t}} \\
2 &= e^{2-2e^{-t}} \\
\ln 2 &= 2 - 2e^{-t} \\
2e^{-t} &= 2 - \ln 2 \\
e^{-t} &= \frac{2-\ln 2}{2} \\
t &= -\ln\left(\frac{2-\ln 2}{2}\right).
\end{aligned}$$

d. [2 points] What happens to the value of the volume of the cell in the long run?

Solution: $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V_0e^{2-2e^{-t}} = V_0e^2$. Hence the volume of the cell $V(t)$ approaches the value V_0e^2.

7. [7 points] Bill has just built a brand new 90,000 L swimming pool. Bill is allergic to chlorine so instead he is using a filtration system to prevent algae from building up in the pool. Algae grows in the pool at a constant rate of 600 kg/day. The filtration system receives a constant supply of 70,000 L/day of water and returns the water to the pool with 6/7ths of the algae removed. Let $A(t)$ be the amount of algae in the pool in kilograms t days after Bill has filled the pool with fresh (algae free) water.

a. [5 points] Write down the differential equation satisfied by $A(t)$. Include the initial condition.

Solution:
$\frac{dA}{dt}$ = Rate in-Rate out. Rate in = 600 kg/day. Rate out=flow rate× concentration× fraction removed= $70,000 \times \frac{A}{90,000} \times 6/7 = \frac{2}{3}A$. Thus $\frac{dA}{dt} = 600 - \frac{2}{3}A$.

$$\frac{dA}{dt} = 600 - \frac{2}{3}A$$

Initial condition: $A(0) = 0$

b. [2 points] Find all the equilibrium solutions of the differential equation.

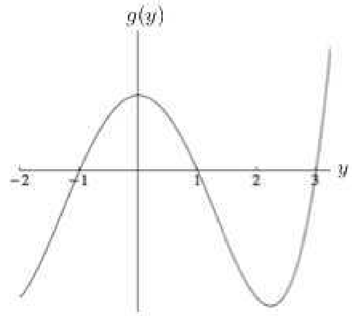
Solution:
We want to solve $\frac{dA}{dt} = 600 - \frac{2}{3}A = 0$. Therefore $A = 900$ is the only equilibrium solution.

8. [4 points] Consider the differential equation $y' = e^y$. Solve the differential equation with initial condition $y(0) = 1$.

Solution:
The equation $\frac{dy}{dx} = e^y$ is separable so we have $e^{-y}dy = dx$. Integrating both sides we get $-e^{-y} = x + c$. Solving the equation we get $y = -\ln(c - x)$. To solve for c we take $y(0) = -\ln(c) = 1$. Therefore $c = \frac{1}{e}$. So the solution is $y = -\ln(\frac{1}{e} - x)$.

2. [12 points]

a. [10 points] Suppose the function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = g(y)$, where the graph of $g(y)$ is shown below:



1.(4 pts) Use inequalities to describe the regions in the y-t plane where the solution curves of the differential equation are strictly increasing.

Solution: $y(t)$ is increasing on $-1 < y < 1$ and $3 < y$ since $\frac{dy}{dt} = g(y) > 0$ in these intervals.

2.(6 pts) Find all equilibrium solutions (if any) to the differential equation for $y(t)$. Classify each one as stable or unstable. If the equation does not have equilibrium solutions, write none.

Solution: $y = -1$ and $y = 3$ are unstable, $y = 1$ is stable.

b. [2 points] Consider the differential equation

$$\frac{dy}{dt} = (2y + 5t)t.$$

Find all equilibrium solutions (if any) to the differential equation for y. If the equation does not have equilibrium solutions, write none.

Solution: None

9. [10 points] Vic is planning to put ladybugs in his garden to eat harmful pests. The ladybug expert at the gardening store claims that the number of ladybugs in his garden can be modeled by the differential equation

$$\frac{dL}{dt} = \frac{L}{20} - \frac{L^2}{100}$$

where L is the number of ladybugs, in hundreds, in Vic's garden, t days after they are introduced.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate their stability.

Solution: To find the equilibrium solutions we set the right hand side of the equation above equal to 0. The resulting equation simplifies to $L(5-L) = 0$. There is then an unstable equilibrium at $L = 0$ and a stable equilibrium at $L = 5$.

b. [2 points] If Vic starts his garden with 50 ladybugs, what will the long term population of ladybugs in his garden be according to the differential equation above?

Solution: The long term population is 500 ladybugs, as the solution to the differential equation above with initial condition $L(0) = .5$ will tend to the stable equilbrium at $L = 5$ as $t \to \infty$.

The long term population is **500 ladybugs**

c. [4 points] For what value of b is the function $L(t) = 5e^{bt}\left(4 + e^{bt}\right)^{-1}$ a solution to this differential equation.

Solution: We can compute that $\dfrac{dL}{dt} = 5be^{bt}\left(4 + e^{bt}\right)^{-1} - 5e^{2bt}\left(4 + e^{bt}\right)^{-2} = \dfrac{20be^{bt}}{(1+e^{bt})^2}$ and $\dfrac{L}{20} - \dfrac{L^2}{100} = \dfrac{5e^{bt}\left(4+e^{bt}\right)^{-1}}{20} - \dfrac{25e^{2bt}\left(4+e^{bt}\right)^{-2}}{100} = \dfrac{e^{bt}}{(1+e^{bt})^2}$. These two expressions will be equal provided that $b = \dfrac{1}{20}$.

$b =$ **1/20**

6. [10 points] At a hospital, a patient is given a drug intravenously at a constant rate of r mg/day as part of a new treatment. The patient's body depletes the drug at a rate proportional to the amount of drug present in his body at that time. Let $M(t)$ be the amount of drug (in mg) in the patient's body t days after the treatment started. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = r - \frac{1}{4}M \quad \text{with} \quad M(0) = 0.$$

a. [7 points] Find a formula for $M(t)$. Your answer should depend on r.

Solution: We use separation of variables

$$\frac{dM}{r - \frac{1}{4}M} = dt.$$

Using u-substition with $u = r - 1/4M, du = -1/4dM$ for the left-hand-side, we anti-differentiate:

$$-4\ln|r - \frac{1}{4}M| = t + C_1.$$

Therefore,

$$\ln|r - \frac{1}{4}M| = -t/4 + C_2$$

and

$$|r - \frac{1}{4}M| = e^{-t/4+C_2} = C_3e^{-t/4}.$$

Therefore

$$1/4M = r - C_3e^{-t/4}$$

and

$$M(t) = 4r - C_4e^{-t/4}.$$

With $M(0) = 0$, we conclude that $C_4 = 4r$, so we get $M(t) = 4r - 4re^{-t/4}$.

b. [1 point] Find all the equilibrium solutions of the differential equation.

Solution: $M = 4r$.

c. [2 points] The treatment's goal is to stabilize in the long run the amount of drug in the patient at a level of 200 mg. At what rate r should the drug be administered?

Solution: You need $4r = 200$, then $r = 50$ mg/day.

4. [5 points] Drake is running for president. Suppose $F(t)$ is the fraction of the total population of the country who supports him t months after he announces he is running. Drake gains supporters at a steady rate of 2% of the **total population** of the country per month, but he also steadily loses 3% of **his supporters** per month. Write a **differential equation** that models $F(t)$.

Solution:

$$\frac{dF}{dt} = 0.02 - 0.03F$$

5. [6 points] Adele is also running for president. Suppose $P(t)$, the total number of supporters she has in millions t days after she announces, is modeled by the differential equation

$$\frac{dP}{dt} = kP(100 - P)$$

with $k > 0$.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate stabilities for each. Make sure your answer is clear.

Solution: The equilibrium $P = 0$ is unstable and the equilibrium $P = 100$ is stable.

b. [2 points] If Adele starts with one million supporters, what is the maximum number of supporters she can get in the long run? You do not need to show your work.

Solution: $100,000,000$ supporters

9. [14 points] An ice cube melts at a rate proportional to its surface area. Let $V(t)$ denote the volume (in cm^3) of the ice cube, and let $x(t)$ denote the length (in cm) of a side of the ice cube t seconds after it begins to melt.

a. [4 points] Write a differential equation for $V(t)$, the ice cube's volume t seconds after it started melting. Your differential equation may contain V, t and an unknown constant k.

Solution: We know that $V = x^3$, so $x = V^{1/3}$. The surface area of the cube is given by $6x^2$. That gives us $\frac{dV}{dt} = 6kx^2$, and substituting x in terms of V, we have $\frac{dV}{dt} = 6kV^{2/3}$.

b. [4 points] The ice cube's initial volume is $V_0 > 0$. Solve the differential equation you found in part (a), finding V in terms of t, k, and V_0.

Solution: Using separation of variables, we have $\frac{dV}{V^{2/3}} = 6kdt$. This gives $3V^{1/3} = 6kt + C$, and $V^{1/3} = 2kt + C$, or $V = (2kt + C)^3$. When $t = 0$, $V = V_0$, which gives us $V_0 = C^3$, so that $C = V_0^{1/3}$. The solution is then $V = (2kt + V_0^{1/3})^3$.

c. [6 points] Graph the volume of the ice cube versus time given $V(0) = V_0$. Be sure to label your axes and any important features of your graph, including the time at which the ice cube has completely melted.

Solution: The vertical intercept is $V = V_0$. The horizontal intercept is $t = -\frac{1}{2k}V_0^{1/3}$.

4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ____________________

ii. Find the differential equation satisfied by $V(t)$. Include initial conditions.

Solution: i) The formula is $V(t) = 64\pi h(t)$.

ii) The differential equation is

$$\frac{dV}{dt} = 2 - kh.$$

So now we can solve $h(t) = \frac{V(t)}{64\pi}$. Substituting in V for h, we get

$$\frac{dV}{dt} = 2 - k\frac{V}{64\pi}$$

with initial condition $V(0) = 0$.

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

Solution: This equation is separable:

$$\frac{dM}{M - 100a} = \frac{1}{100}dt.$$

Integrating, we find $\ln|M - 100a| = \frac{t}{100} + C$. So we get

$$M = Be^{t/100} + 100a.$$

Using the initial conditions, $M(0) = 1000$, so $1000 = B + 100a$. Substituting back in we get

$$M = 100\left((10 - a)e^{t/100} + a\right).$$

7. [14 points] You want to open a savings account to deposit 1000 dollars. Three banks offer the following options:

a. [3 points] Bank A offers its clients a savings account that earns 1.5% per year compounded annually. Define the sequence A_n to be the amount of money in the savings account n years after you deposit your 1000 dollars. Find a formula for A_n.

Solution: $A_n = 1000(1.015)^n$

b. [7 points] Bank B offers its clients a savings account that earns 2% per year compounded annually. At the end of each year, after the bank deposits the interest you earned, it withdraws a 1 dollar service fee from the account. Define the sequence B_n to be the amount of money, right after the service fee deduction, in the savings account n years after you deposit your 1000 dollars. Find B_1, B_2, B_3 and a **closed form** formula for B_n.

Solution:

$$\begin{aligned}
B_1 &= 1000(1.02) - 1 = 1019. \\
B_2 &= (1000(1.02) - 1)(1.02) - 1 = 1000(1.02)^2 - (1 + 1.02) = 1038.38. \\
B_3 &= \big(1000(1.02)^2 - (1 + 1.02)\big)(1.02) - 1 = 1000(1.02)^3 - (1 + 1.02 + 1.02^2) \\
&= 1058.15. \\
&\vdots \\
B_n &= 1000(1.02)^n - (1 + 1.02 + 1.02^2 + \cdots + 1.02^{n-1}) = 1000(1.02)^n - \frac{1 - 1.02^n}{1 - 1.02}
\end{aligned}$$

c. [4 points] Bank C offers its clients a savings account that earns interest continuously at a rate of 1.5% of the current balance per year. At the same time, the bank withdraws a service fee from the account at a rate of 1 dollar per year continuously. Let $M(t)$ be the amount of money in the savings account t years after you deposit your 1000 dollars. Write the differential equation satisfied by $M(t)$. Include initial conditions.

Solution: $\dfrac{dM}{dt} = 0.015M - 1, \qquad M(0) = 1000.$

3. [14 points] A farmer notices that a population of grasshoppers is growing at undesirable levels in his crop. He decides to hire the services of a pest control company. They offer the farmer a pesticide capable of eliminating the grasshoppers at a rate of 1 thousand grasshoppers per week. In the absence of pesticides, it is estimated that the grasshopper population grows at a rate of 20 percent every week. Let $P(t)$ be the number of grasshoppers (in thousands) t weeks after the pesticide is applied to the crop. Then $P(t)$ satisfies

$$\frac{dP}{dt} = \frac{P}{5} - 1.$$

Suppose there are P_0 thousand grasshoppers in the crop at the time the pesticide is applied in the crop.

a. [8 points] Find a formula for $P(t)$ in terms of t and P_0.

Solution:

$$\begin{aligned} \frac{dP}{dt} &= \frac{P}{5} - 1. \\ \frac{dP}{dt} &= \frac{1}{5}(P-5) \\ \frac{dP}{P-5} &= \frac{1}{5}dt \\ \ln|P-5| &= \frac{1}{5}t + C \\ P - 5 &= Be^{\frac{1}{5}t} \quad P(0) = P_0 = 5 + B \quad B = P_0 - 5. \\ P(t) &= 5 + (P_0 - 5)e^{\frac{1}{5}t}. \end{aligned}$$

b. [3 points] Does the differential equation have any equilibrium solutions? List each equilibrium solution and determine whether it is stable or unstable. **Justify your answer**.

Solution: Equilibrium solutions: $P(t) = 5$. The equilibrium is unstable since for $P_0 > 5$ $P(t)$ increases and for $P_0 < 5$ $P(t)$ decreases.

c. [3 points] Does the effectiveness of the pesticide depend on P_0? That is, is the pesticide guaranteed to eliminate the grasshopper population regardless of the value of P_0, or are there some values of P_0 for which the grasshoppers will survive? If so, determine these values of P_0.

Solution: The pesticide is effective if $P_0 < 5$ and ineffective if $P_0 \geq 5$.