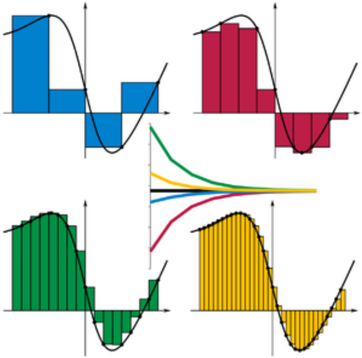
# Introduction

Suppose we have a table of values of the velocity and we want to see how far the position moved.
What can we do? Problem 2

Four of the Riemann summation methods for approximating the area under curves. Right (Blue) and left (yellow) methods make the approximation using the right and left endpoints of each subinterval, respectively. Maximum (green) and minimum (red) methods make the approximation using the largest and smallest endpoint values of each subinterval, respectively.



# Sigma notation

$\Delta t = \frac{b-a}{n}$
Let $t_0, t_1, ..., t_n$ be the endpoints of the subdivisions.
right hand sum $= f(t_1)\Delta t + f(t_2)\Delta t + ... + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$
left hand sum $= f(t_0)\Delta t + f(t_1)\Delta t + ... + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t$

Suppose $f$ is continuous for $a \leq t \leq b$. The definite integral of $f$ from $a$ to $b$, written

$$\int_a^b f(t)dt$$

is the limit of the left-hand or right-hand sums with $n$ subdivisions of $a \leq t \leq b$ as $n$ gets aribitrarily large.
So

$$\int_a^b f(t)dt = \lim_{n\to\infty}(\sum_{i=0}^{n-1} f(t_i)\Delta t)$$

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Each of theses sums is called a Riemann sum. $f$ is called the integrand, and $a$ and $b$ are the limits of integration.

When $f(x) \geq 0$ and $a < b$
Area under the graph of $f$ and above $x$-axis between $a$ and $b$ is

$$\int_a^b f(x)dx$$

When $f(x)$ is positive for some $x$ values and negative for others, and $a < b$:
$\int_a^b f(x)dx$ is the sum of the areas above the $x$-axis, counted positively, and the areas below the $x$-axis, counted negatively.

A general Riemann sum for $f$ on the interval $[a, b]$ is a sum of the form

$$\sum_{i=1}^{n} f(c_i)\Delta t_i$$

where $a = t_0 < t_1 < ... < t_n = b$, and, for $i = 1, ..., n$, $\Delta t_i = t_i - t_{i-1}$, and $t_{i-1} \leq c_i \leq t_i$

Written as

$$\int_a^b f(x)dx$$

or

$$\int_a^b f(t)dt$$

You can think $dx$ (or $dt$) as infinitesmall small bit of $x$ ($t$) multiplied by $f(x)$ ($f(t)$). $dx$ has units.
For example take

$$\int_a^b v(t)dt$$

where $v(t)$ is velocity then $dt$ can be thought to measure in seconds. Note that $t$ and $dt$ have the same units. so $v(t)$ has units $\frac{meters}{secs}$ and $dt$ has units $secs$, So $\int_a^b v(t)dt$ has units $\frac{meters}{secs}(secs) = meters$

# Antiderivatives

If the derivative of $F$ is $f$, we call $F$ the *antiderivative* of $f$.
Example $sin(x^2)$ is the antiderivative of $cos(x^2)2x$
Note that $sin(x^2) + 1$, $sin(x^2) + 2$ etc are also antiderivative of $cos(x^2)2x$.
In fact any function of the form $sin(x^2) + C$ is an antiderivative.
We say $f(x)$ has a family of antiderivatives.

# Fundamental Theorem of Calculus

If $f$ is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

# Properties

If $f$ is a continuous function and $a, b, c$ are any numbers then:

$$\int_b^a f(t)dt = -\int_a^b f(t)dt$$

$$\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt$$

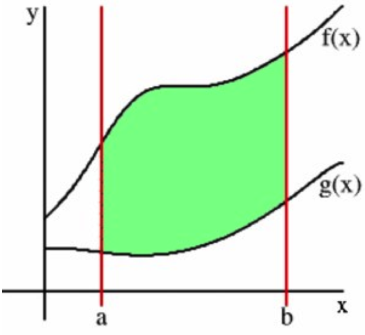
Let $f$ and $g$ be continuous functions and let $c$ be constant:

$$\int_a^b (f(t) \pm g(t))dt = \int_a^b f(t)dt \pm \int_a^b g(t)dt$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt$$

If $f(t)$ lies above $g(t)$ and $a \leq t \leq b$ then:

$$\int_a^b (f(t) - g(t))dt = \text{Area between } f \text{ and } g \text{ for } a \leq t \leq b$$

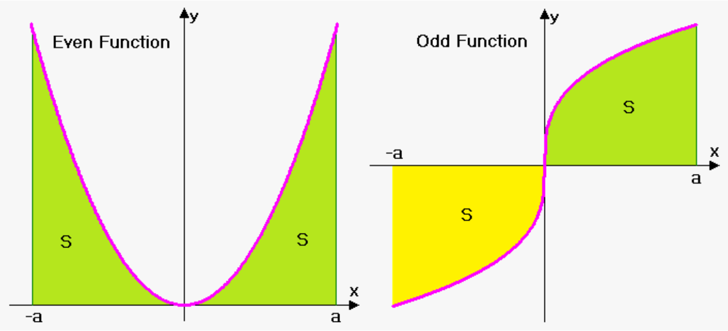


If $f$ is even, then:

$$\int_{-a}^{a} f(t)dt = 2\int_{0}^{a} f(t)dt$$

If $g$ is odd, then:

$$\int_{-a}^{a} g(t)dt = 0$$



Let $f$ and $g$ be continuous functions.
If $m \leq f(t) \leq M$ for $a \leq t \leq b$, then

$$m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$$

If $f(t) \leq g(t)$ for $a \leq t \leq b$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt$$

Average value of $f$ from $a$ to $b$ is

$$\frac{1}{b-a}\int_a^b f(t)dt$$

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We say $f(x)$ has a family of antiderivatives.

If $F'(x) = 0$ on an interval, then $F(x) = C$ on this interval, for some constant $C$.
If $F$ and $G$ are both antiderivatives of $f$ on an interval, then
$G(x) = F(x) + C$

# Notation

Since all antiderivatives of $f(x)$ are of the form $F(x) + C$ then we call the **indefinite integral**

$$\int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx \text{ is a number}$$

$$\int f(x)dx \text{ is a family of functions}$$

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{x} dx = \ln(|x|) + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

$$\int cf(x)dx = c\int f(x)dx$$

# Second Fundamental Theorem of Calculus

If $f$ is continuous on the interval and if $a$ is any number in that interval, then the function $F$ defined on the interval as follows is an antiderivative of $f$:

$$F(x) = \int_a^x f(t)dt$$