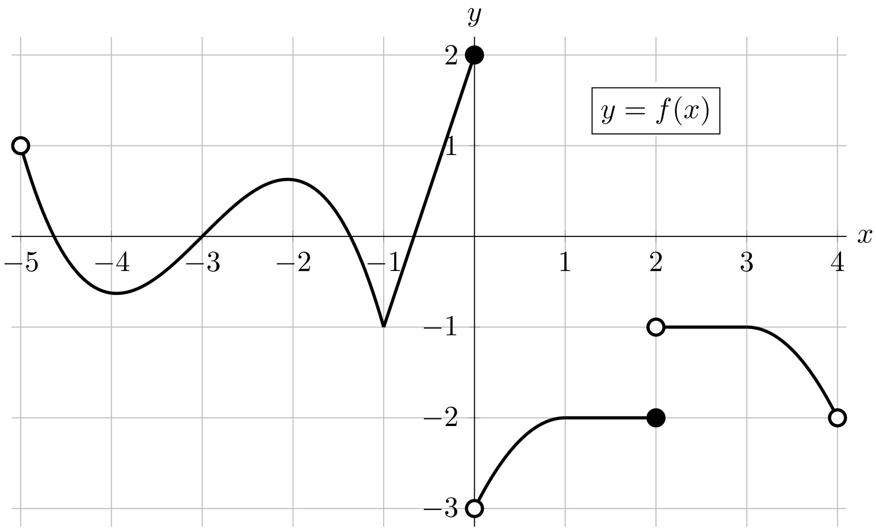
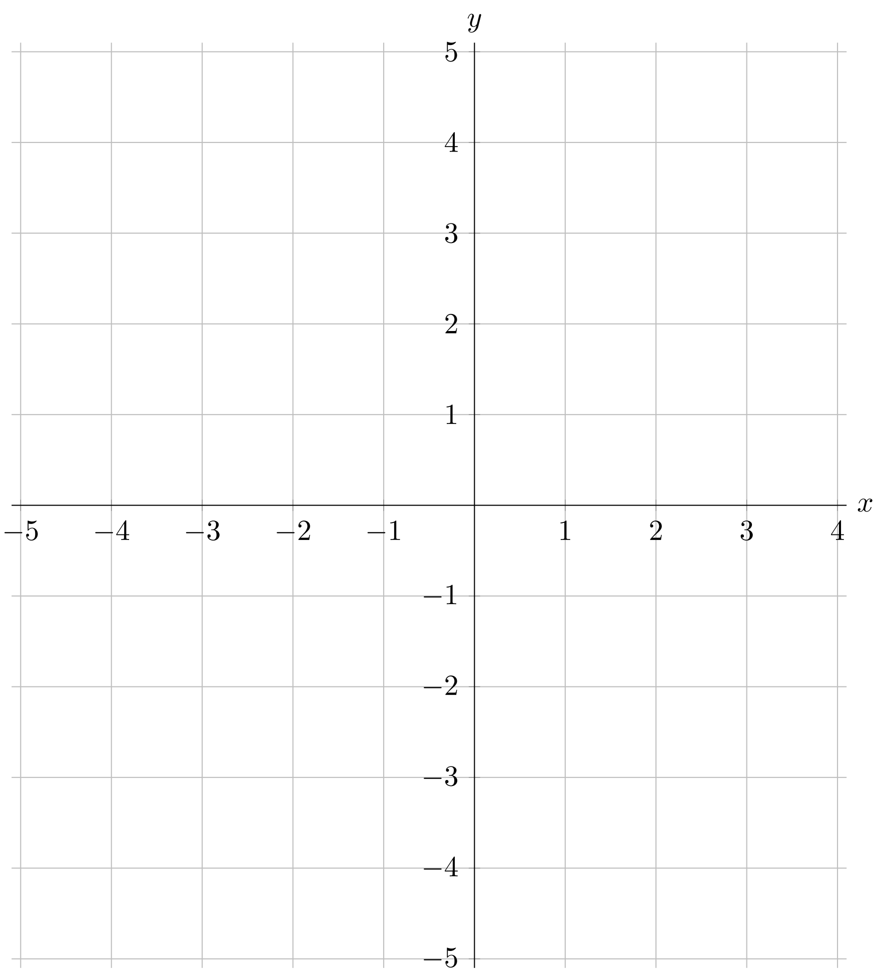
4. [8 points] The graph of a function f is shown below.



On the axes below, sketch a graph of $f'(x)$ (the derivative of the function $f(x)$) on the interval $-5 < x < 4$. Be sure that you pay close attention to each of the following:

- where f' is defined
- the value of $f'(x)$ near each of $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$
- the sign of f'
- where f' is increasing/decreasing/constant



2. [14 points] Suppose $f(x)$ is a function defined for all real numbers whose derivative and second derivative are given by

$$f'(x) = (x-4)^3(x+2)^{2/3} \quad \text{and} \quad f''(x) = \frac{(11x+10)(x-4)^2}{3(x+2)^{1/3}}.$$

a. [7 points] Find all critical points of $f(x)$ and all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Answer: Critical point(s) at $x =$ ________________

Local max(es) at $x =$ ____________ Local min(s) at $x =$ ____________

b. [7 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Answer: Inflection point(s) at $x =$ ________________

9. [14 points]

a. [8 points] Consider functions f satisfying <u>all</u> of the following conditions:

- $f(x)$ is differentiable on the interval $0 < x < 8$.
- The critical points of $f(x)$ in the interval $0 < x < 8$ are $x = 2, 4$, and 6. ($f(x)$ has no other critical points in this interval.)
- The table below shows some values of $f(x)$ and of its derivative $f'(x)$.

x	1	3	5	7
$f(x)$	3	6	11	0
$f'(x)$	-1	?	?	-1

For each of the statements below, decide whether the statement is true for ALL functions f satisfying all of the conditions described above, for SOME of these functions f, or for NONE of these functions f. Circle the <u>one</u> correct choice for each statement.

(i) $f(x)$ has a local minimum at $x = 2$.

ALL SOME NONE

(ii) $f'(3) > 0$.

ALL SOME NONE

(iii) $f(x)$ has a local maximum at $x = 4$.

ALL SOME NONE

(iv) There is exactly one value of a with $3 < a < 7$ such that $f(x)$ has a local maximum at $x = a$.

ALL SOME NONE

b. [6 points] Consider functions g satisfying <u>all</u> of the following conditions:

- $g(z)$ and $g'(z)$ are differentiable on the interval $12 < z < 18$.
- The critical points of $g(z)$ in the interval $12 < z < 18$ are $z = 14$ and $z = 16$. ($g(z)$ has no other critical points in this interval.)
- The table below shows some values of $g(z)$ and of its <u>second derivative</u> $g''(z)$.

z	13	14	15	16	17
$g(z)$	8	?	6	?	2
$g''(z)$	?	-1	?	0	?

For each of the statements below, decide whether the statement is true for ALL functions g satisfying all of the conditions described above, for SOME of these functions g, or for NONE of these functions g. Circle the <u>one</u> correct choice for each statement.

(i) $g(z)$ has a local extremum at $z = 14$.

ALL SOME NONE

(ii) $g'(15) > 0$.

ALL SOME NONE

(iii) $g(z)$ has an inflection point at $z = 16$.

ALL SOME NONE

2. [10 points] Let a be a constant with $a > 1$.
A function $w(x)$ and its derivative $w'(x)$ are given below.

$$w(x) = a + \frac{x}{x^2 + a^2} \quad \text{and} \quad w'(x) = \frac{-(x-a)(x+a)}{(x^2+a^2)^2}.$$

a. [5 points] Find and classify the local extrema of $w(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate.

Answer: Local min(s) at $x =$ ______________________

Answer: Local max(es) at $x =$ ______________________

b. [5 points] Find the global extrema of $w(x)$ on the interval $[1, \infty)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found the global extrema. For each answer blank, write NONE if appropriate.

Answer: Global min(s) at $x =$ ______________________

Answer: Global max(es) at $x =$ ______________________

8. [11 points] A function $g(t)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Answer: Critical point(s) at $t =$ ____________________

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Answer: Global max(es) at $t =$ __________ Global min(s) at $t =$ __________

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

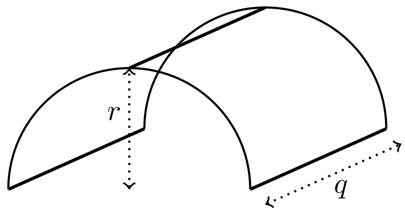
Answer: Global max(es) at $t =$ __________ Global min(s) at $t =$ __________

c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer blank, write NONE if appropriate. You do not need to justify your answers.

Answer: Critical point(s) at $t =$ _______________

Answer: Inflection point(s) at $t =$ _______________

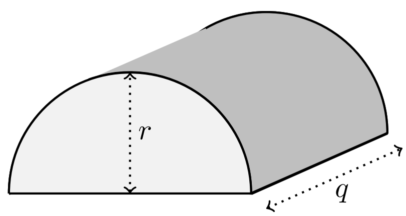
3. [9 points] Duncan's person is making him a new tent in the shape of half a cylinder. She plans to use wire to make the tent frame. This will consist of two semicircles of radius r (measured in inches) attached to three pieces of wire of length q (also measured in inches), as shown in the diagram below. She has 72 inches of wire to use for this.



a. [4 points] Find a formula for r in terms of q.

Answer: $r =$ ____________________

b. [2 points] Let $V(q)$ be the volume (in cubic inches) of the space inside the tent after the fabric is added, given that the total length of wire is 72 inches and the length of the tent is q inches. (Recall that the tent shape is half of a cylinder.) Find a formula for $V(q)$. The variable r should not appear in your answer.
(Note: This is the function that Duncan's person would use to find the value of q that maximizes the volume of the tent, but you should not do the optimization in this case.)



Answer: $V(q) =$ ____________________

c. [3 points] In the context of this problem, what is the domain of $V(q)$?

Answer: ____________________

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Answer: $v(t) =$ ____________________

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ ____________________

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Answer: Max vertical acceleration: ____________________

9. [10 points] Our friend Oren, the Math 115 student, wants to minimize how long it will take him to complete his upcoming web homework assignment. Before starting the assignment, he buys a cup of tea containing 55 milligrams of caffeine.

Let $H(x)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes x milligrams of caffeine. For $10 \leq x \leq 55$

$$H(x) = \frac{1}{120}x^2 - \frac{4}{3}x + 20\ln(x)\,.$$

Instead of immediately starting the assignment, he solves a calculus problem to determine how much caffeine he should consume.

a. [8 points] Find all the values of x at which $H(x)$ attains global extrema on the interval $10 \leq x \leq 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

(For each answer blank below, write NONE in the answer blank if appropriate.)

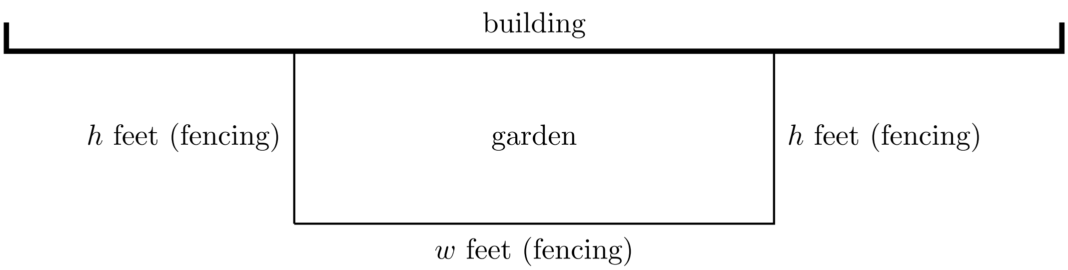
Answer: global min(s) at $x =$ __________________

Answer: global max(es) at $x =$ __________________

b. [2 points] Assuming Oren consumes at least 10 milligrams and at most 55 milligrams of caffeine, what is the shortest amount of time it could take for him to finish his assignment? *Remember to include units.*

Answer: __________________

4. [12 points] Researchers are constructing a rectangular garden adjacent to their building. The garden will be bounded by the building on one side and by a fence on the other three sides. (See diagram below.) The fencing will cost them \$5 per linear foot. In addition, they will also need topsoil to cover the entire area of the garden. The topsoil will cost \$4 per square foot of the garden's area.
Assume the building is wider than any garden the researchers could afford to build.



a. [5 points] Suppose the garden is w feet wide and extends h feet from the building, as shown in the diagram above. Assume it costs the researchers a total of \$250 for the fencing and topsoil to construct this garden. Find a formula for w in terms of h.

Answer: $w =$ ______________________

b. [3 points] Let $A(h)$ be the total area (in square feet) of the garden if it costs \$250 and extends h feet from the building, as shown above. Find a formula for the function $A(h)$. The variable w should not appear in your answer.

(Note that $A(h)$ is the function one would use to find the value of h maximizing the area. You should not do the optimization in this case.)

Answer: $A(h) =$ ______________________

c. [4 points] In the context of this problem, what is the domain of $A(h)$?

Answer: ______________________

4. [11 points] Elphaba has found a corrupt prison guard, Mert, to sell her metal piping to use to dig a tunnel out of the prison. Mert can sell Elphaba steel piping and copper piping, and he provides the following information.

- The number of kilograms (kg) of soil that Elphaba can dig with steel piping is proportional to the number of centimeters (cm) of steel piping that she buys. She can dig 50 kg of soil per cm of steel piping, and her cost (in dollars) of buying x cm of steel piping is given by $A(x) = x^2 + x$.
- The number of kilograms (kg) of soil that Elphaba can dig with copper piping is proportional to the number of centimeters (cm) of copper piping that she buys. She can dig 30 kg of soil per cm of copper piping, and her cost (in dollars) of buying y cm of copper piping is given by $B(y) = 2y$.

a. [1 point] How many kilograms of soil can Elphaba dig with x cm of steel piping?

Answer: ________________________

For parts (b)-(d) below, suppose Elphaba buys w cm of steel piping and k cm of copper piping and that this is exactly the right amount of piping so that she can dig through 2700 kg of soil to dig her escape tunnel.

b. [3 points] Write a formula for k in terms of w.

Answer: $k =$ ________________________

c. [4 points] Let $T(w)$ be the total cost (in dollars) of all the piping Elphaba buys to dig her escape tunnel. Find a formula for the function $T(w)$. The variable k and the function names A and B should not appear in your answer.
(Note that $T(w)$ is the function one would use to minimize Elphaba's costs. You should not do the optimization in this case.)

Answer: $T(w) =$ ________________________

d. [3 points] What is the domain of $T(w)$ in the context of this problem?

Answer: ________________________