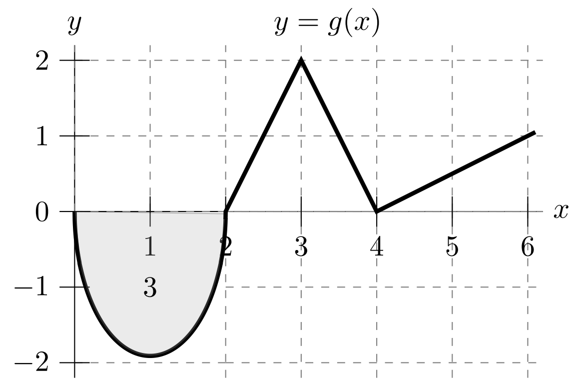
6. [10 points]

A portion of the graph of a continuous function $g(x)$ is shown on the right. Assume that the area of the shaded region is 3 (as indicated on the graph), and note that $g(x)$ is piecewise linear for $2 < x < 6$.



For each of parts **a.-e.** below, find the value of the given quantity. If there is not enough information provided to find the value, write NOT ENOUGH INFO. If the value does not exist, write DOES NOT EXIST.

Remember to show your work.

a. [2 points] Find $\displaystyle\int_0^6 g(x)dx$.

Answer: ____________________

b. [2 points] Find $\displaystyle\int_0^2 (5 - 4g(x))dx$.

Answer: ____________________

c. [2 points] Suppose $C(x) = \ln(g(x))$. Find $C'(2.5)$.

Answer: ____________________

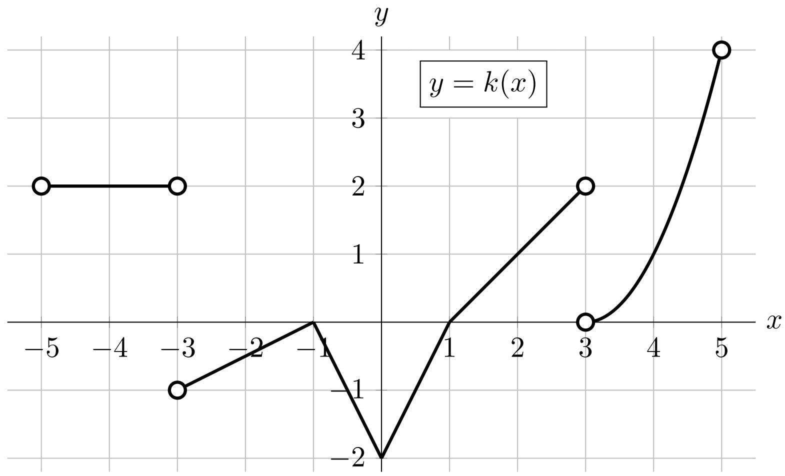
d. [2 points] Find the average value of $g(x)$ on the interval $0 \leq x \leq 4$.

Answer: ____________________

e. [2 points] Find $\displaystyle\int_2^4 (g(x+2) - g(x-2))dx$.

Answer: ____________________

11. [10 points] The graph of a portion of $y = k(x)$ is shown below. Note that for $3 < x < 5$, the graph of $k(x)$ is a portion of the graph obtained by shifting $y = x^2$ three units to the right.



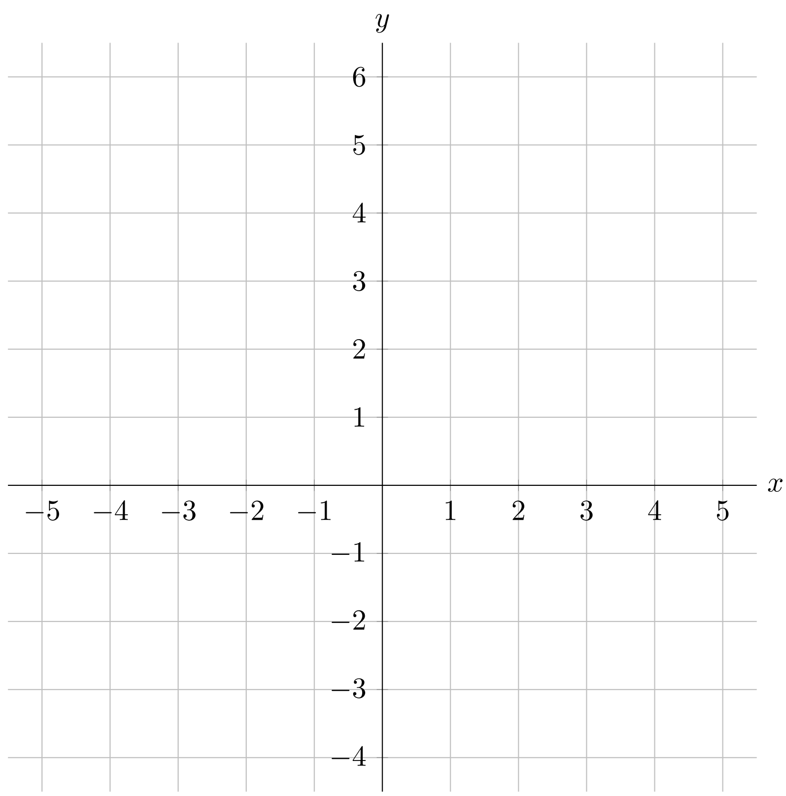
Let $K(x)$ be the continuous antiderivative of $k(x)$ passing through the point $(-1, 1)$.

a. [5 points] Use the graph to complete the table below with the exact values of $K(x)$.

x	-5	-3	-1	1	3	5
$K(x)$						

b. [5 points] On the axes below, sketch a detailed graph of $y = K(x)$ for $-5 < x < 5$. Be sure that you pay close attention to each of the following:

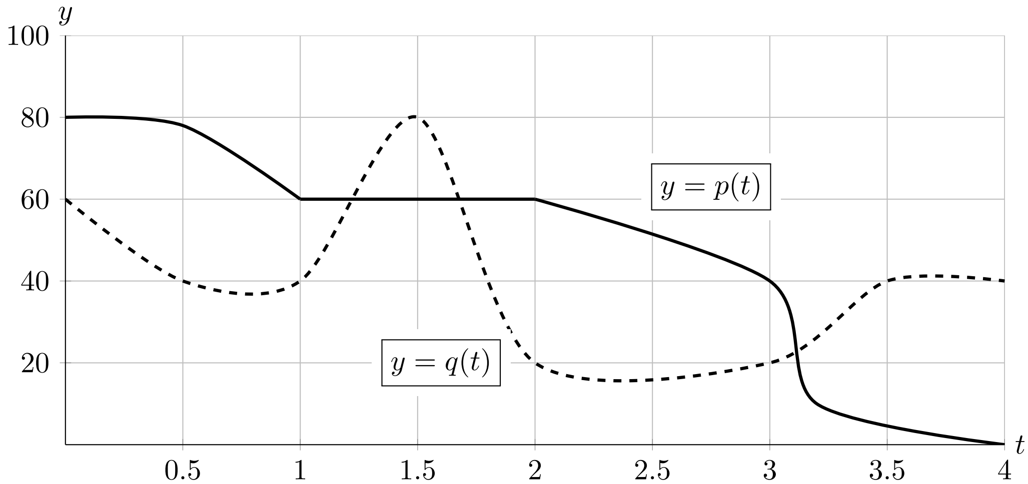
- where $K(x)$ is and is not differentiable,
- the values of $K(x)$ you found in the table above,
- where $K(x)$ is increasing/decreasing/constant, and the concavity of $K(x)$.



This problem continues the investigation of Xanthippe's donuts.

4. [10 points] For your convenience, the graphs of $p(t)$ and $q(t)$ are reprinted below. Recall:

- The rate, in donuts per hour, at which Xanthippe makes donuts t hours after 7 am is modeled by the function $p(t)$.
- The rate, in donuts per hour, at which customers purchase donuts t hours after 7 am is modeled by the function $q(t)$.
- Assume that at 7 am, Xanthippe begins with no donuts in stock.



a. [4 points] Estimate the total number of donuts produced by 10 am using a right-hand Riemann sum with two equal subintervals. Be sure to write down all the terms in your sum. Is your answer an underestimate or overestimate?

Answer: donuts produced by 10 am ≈ ____________________

This is an (circle one) OVERESTIMATE UNDERESTIMATE

b. [4 points] The number of donuts in stock t hours after 7 am is modeled by the function $s(t)$. Estimate the t-values for all critical points of $s(t)$ in the interval $0 < t < 4$, and estimate all values of t in the interval $0 < t < 4$ at which $s(t)$ has a local extremum. For each answer blank write NONE if appropriate. You do <u>not</u> need to justify your answers.

Answer: Critical point(s) at $t =$ ____________________

Local max(es) at $t =$ ____________ Local min(s) at $t =$ ____________

c. [2 points] At what time is the number of donuts that Xanthippe has in stock the greatest? Round your answer to the nearest half hour. You do <u>not</u> need to justify your answer.

Answer: ____________________

1. [11 points] At a recent UM football game, a football scientist was measuring the excitement density, $E(x)$, in cheers per foot, in a one hundred foot row of the football stadium where x is the distance in feet from the beginning of the row. He took measurements every twenty feet and the data is recorded in this table.

x	0	20	40	60	80	100
$E(x)$	30	24	19	16	13	7

Assume for this problem that $E(x)$ is a decreasing function for $0 \leq x \leq 100$.

a. [6 points] Write a right sum and a left sum which approximate the total cheers in the row. Be sure to write all of the terms for each sum.

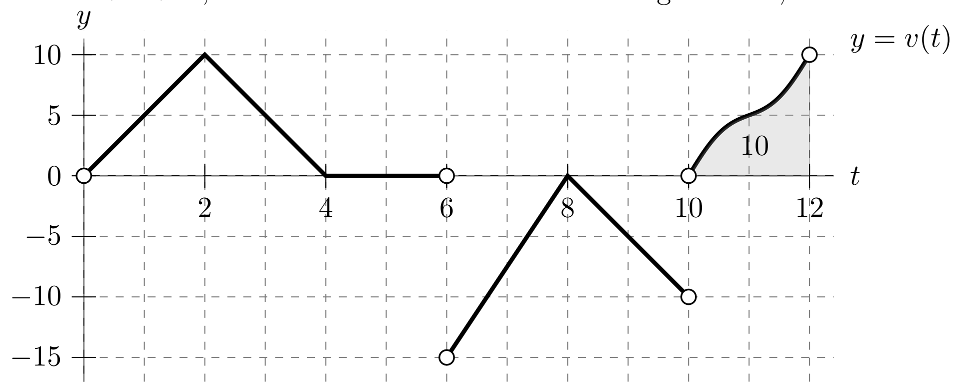
b. [2 points] Indicate whether the right and left sums are overestimates or underestimates for the total number of cheers in the row.

The right sum is an **overestimate** **underestimate**

The left sum is an **overestimate** **underestimate**

c. [3 points] How many measurements must the scientist take to guarantee that the left sum approximates the total number of cheers in the row within 5 cheers of the actual number?

4. [15 points] Elana goes on an amusement park ride that moves straight up and down. Let $v(t)$ model Elana's velocity (in meters/second) t seconds after the ride begins (where $v(t)$ is positive when the ride is moving upwards, and negative when the ride is moving downwards). A graph of $v(t)$ for $0 < t < 12$ is shown below. Assume that $v(t)$ is piecewise linear for $0 < t < 6$ and $6 < t < 10$, and that the area of the shaded region is 10, as indicated on the graph.



a. [4 points] Write an integral that gives Elana's average velocity, in meters/second, from 2 seconds into the ride until 4 seconds into the ride. Then compute the exact value of this integral.

Answer: ____________________ = __________

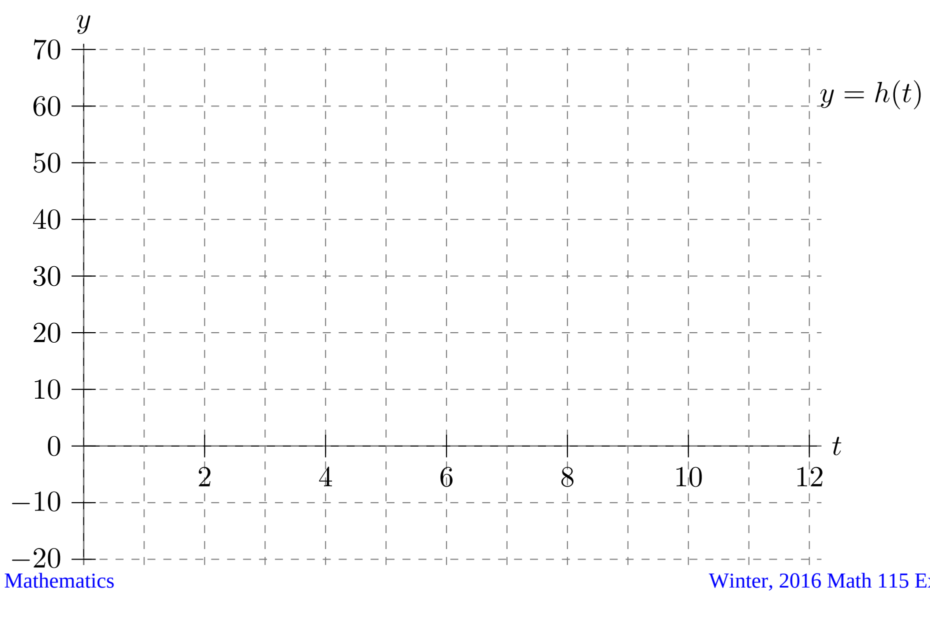
Let $h(t)$ be Elana's height (in meters) above the ground t seconds after the ride begins. Assume that h is continuous, and suppose Elana is at a height of 10 meters above the ground when the ride begins.

b. [6 points] Fill in the exact values of $h(t)$ in the table below.

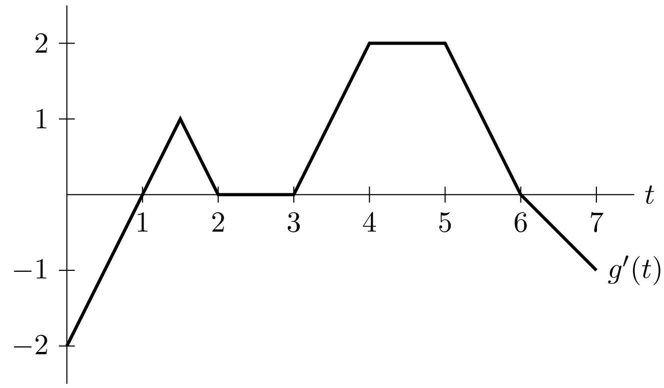
t	0	2	4	6	8	10	12
$h(t)$							

c. [5 points] Using your work from part **b.**, sketch a detailed graph of $h(t)$ for $0 < t < 12$. In your sketch, be sure that you pay close attention to each of the following:

- where h is increasing, decreasing, or constant
- the values of $h(t)$ you found in part **b.** above
- where h is/is not differentiable
- the concavity of the graph of $y = h(t)$



3. [12 points] The function $g(t)$ is the volume of water in the town water tank, in thousands of gallons, t hours after 8 A.M. A graph of $g'(t)$, the **derivative** of $g(t)$, is shown below. Note that $g'(t)$ is a piecewise-linear function.



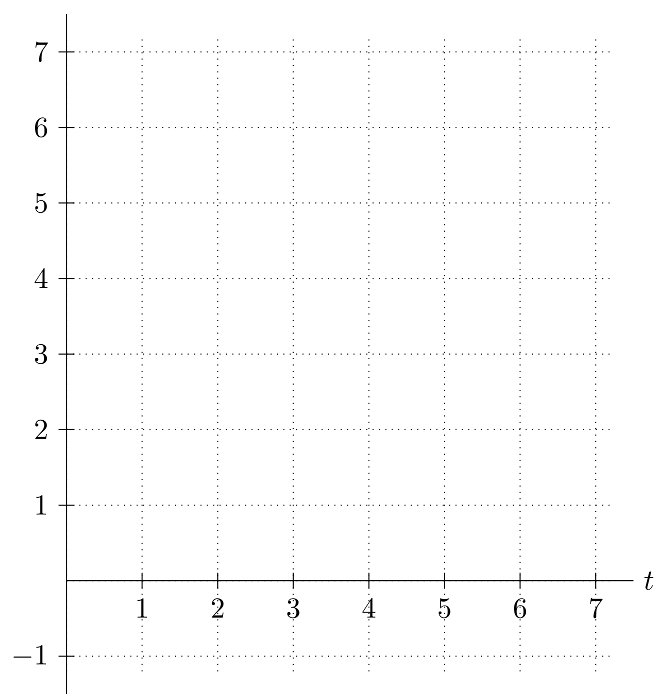
a. [4 points] Write an integral which represents the average rate of change, in thousands of gallons per hour, of the volume of water in the tank between 9 A.M. and 1 P.M. Compute the exact value of this integral.

b. [2 points] At what time does the tank have the most water in it? At what time does it have the least water?

Answer: The tank has the most water in it at ____________________.

The tank has the least water in it at ____________________.

c. [6 points] Suppose that $g(3) = 1$. Sketch a detailed graph of $g(t)$ and give both coordinates of the point on the graph at $t = 7$.



5. [10 points] The table below gives several values of a function $q(u)$ and its first and second derivatives. Assume that all of $q(u)$, $q'(u)$, and $q''(u)$ are defined and continuous for all real numbers u.

u	0	1	2	3	4	5	6
$q(u)$	30	23	19	20	24	25	24
$q'(u)$	0	-6	-2	1	3	1	-2
$q''(u)$	-9	5	4	3	2	-5	0

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

a. [2 points] Compute $\int_5^2 q''(t)\,dt$.

Answer: $\int_5^2 q''(t)\,dt =$ ____________________

b. [2 points] Compute $\int_1^5 (-2q''(u) + 2u)\,du$.

Answer: $\int_1^5 (-2q''(u) + 2u)\,du =$ ____________________

c. [2 points] Suppose that $q(u)$ is an even function. Compute $\int_{-5}^5 q(u)\,du$.

Answer: $\int_{-5}^5 q(u)\,du =$ ____________________

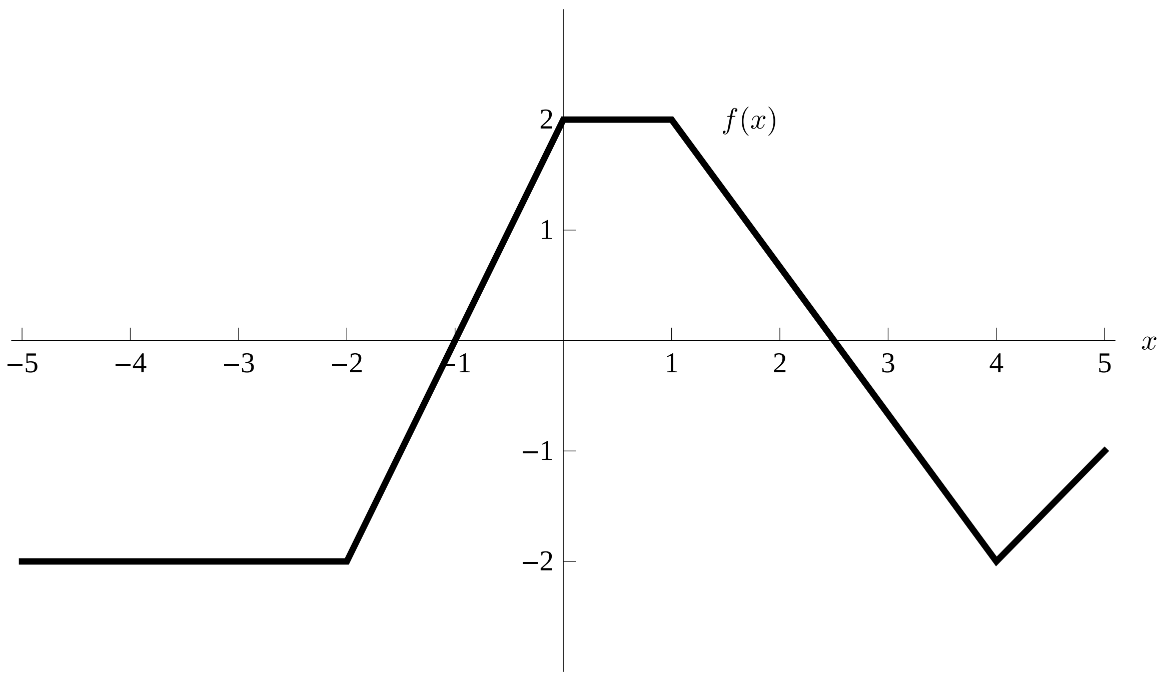
d. [2 points] Suppose that $q(u)$ is an even function. Compute $\int_{-5}^5 \left(q'(u) + 7\right)\,du$.

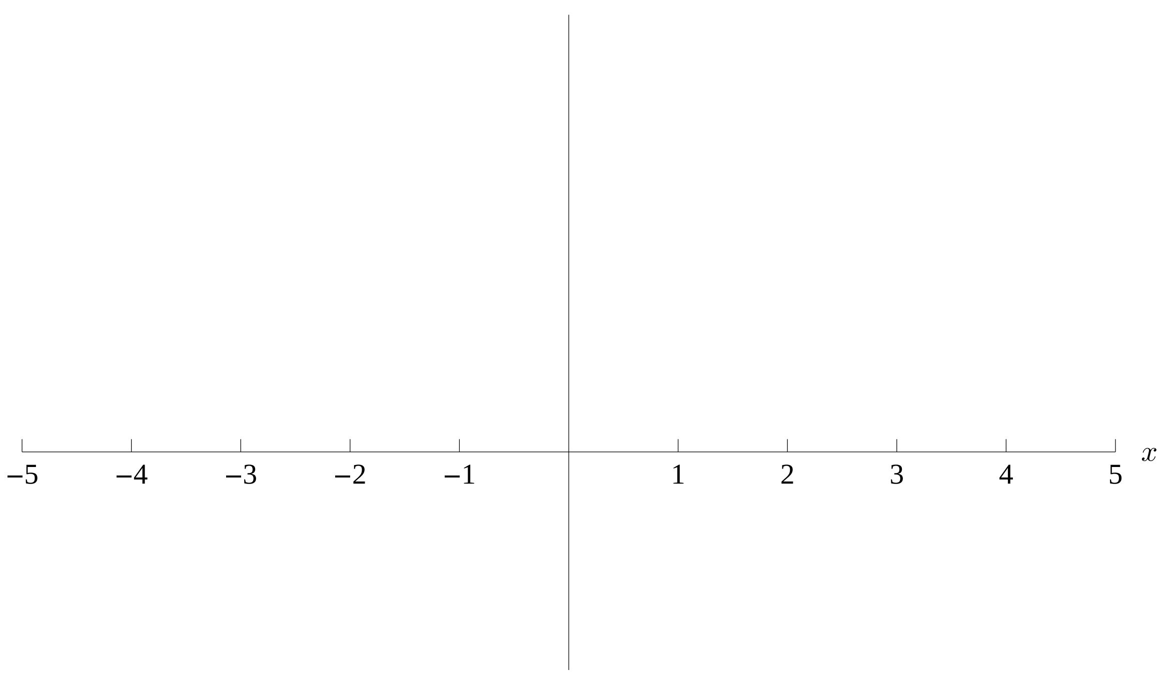
Answer: $\int_{-5}^5 \left(q'(u) + 7\right)\,du =$ ____________________

e. [2 points] Compute the average value of $-5q'(u)$ on the interval $[1, 4]$.

Answer: ____________________

5. [10 points] The graph of a piecewise linear function $f(x)$ is shown below. On the axes provided, sketch a well-labeled graph of an antiderivative $F(x)$ of $f(x)$ satisfying $F(0) = -1$. Be sure to make the concavity of F clear and to label the y-coordinates of the local minima and maxima of F and the y-coordinates of F at $x = 5$ and $x = -5$.





5. [5 points] After hearing of the *Illumisqati* activities from Erin and Elphaba, the Police storm the King's farmhouse and find ample evidence to convict him of kidnapping. However, since he is the King, charges can only be brought against him if the Police can show proficiency in mathematics. Help them by doing the following problem.

For c a constant, consider the function
$$B(u) = \arctan(u^c + 7).$$
Use the limit definition of the derivative to write an explicit expression for $B'(3)$.
Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.

Answer: $B'(3) =$

6. [6 points] Recall the following definitions:

- A function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Compute each of the integrals below. If not enough information is provided to answer the question, write NOT ENOUGH INFORMATION.

a. [2 points] Suppose g is a differentiable function on $(-\infty, \infty)$ and g' (the **derivative** of g) is a continuous odd function with $g(3) = 2$ and $g(7) = 9$. Find $\displaystyle\int_{-3}^{7} g'(x)\,dx$.

Answer: $\displaystyle\int_{-3}^{7} g'(x)\,dx =$ ________________

b. [2 points] Suppose that q is a continuous and even function on $(-\infty, \infty)$ and that $\displaystyle\int_{0}^{5} q(x)\,dx = -4$. Find $\displaystyle\int_{-5}^{5} (3q(x) + 7)\,dx$.

Answer: $\displaystyle\int_{-5}^{5} (3q(x) + 7)\,dx =$ ________________

c. [2 points] Let $h(x) = \ln x$ and suppose p is a differentiable function on $(-\infty, \infty)$ with $p(4) = 7$. Find $\displaystyle\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx$.

Answer: $\displaystyle\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx =$ ________________

8. [11 points] Let $W(t)$ be the temperature, in degrees Fahrenheit, of a cake t minutes after it is put in the oven. Assume $W(10) = 220$.

a. [3 points] Give a practical interpretation of the statement $\int_5^{10} W'(t)dt = 120$.

b. [3 points] Give a practical interpretation of the statement $\frac{1}{2}\int_3^5 W(t)dt = 80$.

c. [3 points] Write a single mathematical equation describing the following statement: The average temperature of the cake over the first five minutes in the oven is the same as its temperature after three minutes in the oven.

d. [2 points] Assuming all of the above statements in (a)-(c) are true, what will the temperature of the cake be five minutes after it is put in the oven?

1. [10 points] The table below gives several values of a function $f(x)$ and its derivative. Assume that both $f(x)$ and $f'(x)$ are defined and differentiable for all x.

x	0	1	2	3	4	5	6
$f(x)$	0	3	4	2	-1	-3	5
$f'(x)$	4	2	-1	-5	-2	7	9
$f''(x)$	-1	-3	-5	0	4	3	1

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE

a. [2 points] Find $\int_0^4 f''(x)\,dx$.

Answer: $\int_0^4 f''(x)\,dx =$ ________________

b. [2 points] Find $\int_2^5 (3f(x)+1)\,dx$.

Answer: $\int_2^5 (3f(x)+1)\,dx =$ ________________

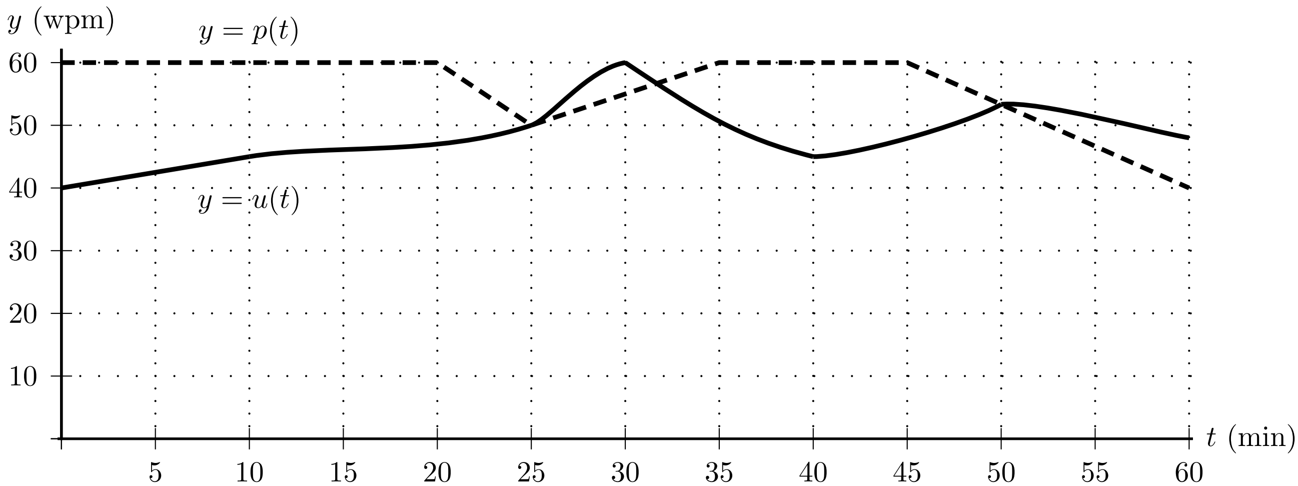
c. [3 points] Find the average value of $4f'(x)+x$ on the interval $[1,6]$.

Answer: ________________

d. [3 points] Assuming that $f(x)$ is an odd function, find $\int_{-3}^3 f(x)\,dx$ and $\int_{-3}^3 f'(x)\,dx$.

Answer: $\int_{-3}^3 f(x)\,dx =$ ________________ and $\int_{-3}^3 f'(x)\,dx =$ ________________

7. [10 points] A history professor gives a 60 minute lecture, while one eager undergraduate student takes notes by typing what the professor says, word for word. Unfortunately, the student cannot always type as quickly as the professor is speaking.
Functions p and u are defined as follows. When t minutes have passed since the start of the lecture, the professor is speaking at a rate of $p(t)$ words per minute (wpm) while the undergraduate student is typing at a rate of $u(t)$ words per minute (wpm). Shown below are graphs of $y = p(t)$ (dashed) and $y = u(t)$ (solid).



a. [2 points] How many minutes after the start of the lecture is the student typing most quickly?

Answer: ____________________

b. [3 points] Write a definite integral equal to the number of words the student types between the start of the lecture and the time the professor reaches the 600th **word** of the lecture. You do not need to evaluate the integral.

Answer: ____________________

c. [3 points] How many minutes after the start of the lecture is the student furthest behind in typing up the lecture? (In other words, after how many minutes is the difference between the total number of words the professor has spoken and the total number of words the student has typed the greatest?)

Answer: ____________________

d. [2 points] What is the average rate, in words per minute, at which the professor is speaking between $t = 40$ and $t = 60$?

Answer: ____________________

7. [10 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

Recall the following definitions:

A function f is *even* if $f(-x) = f(x)$ for all x.

A function f is *odd* if $f(-x) = -f(x)$ for all x.

a. [2 points] If $f(x)$ is an odd function and the tangent line to the graph of $f(x)$ at $x = 2$ is $y = 4(x-2)+7$, then the tangent line to the graph of $f(x)$ at $x = -2$ is $y = -4(x+2)-7$.

True False

b. [2 points] If $g''(x) = 2^x(x-4)(x+5)^2$, then $g(x)$ has inflection points at $x = 4$ and $x = -5$.

True False

c. [2 points] If $h(x)$ is an even function and $\int_{-3}^{8} h(x)\,dx = 17$, then $\int_{-8}^{3} h(x)\,dx = 17$.

True False

d. [2 points] If $\int_{3}^{7} p(t)\,dt = -5$, then $\int_{-1}^{3} p(t-4)\,dt = -5$.

True False

e. [2 points] If $f(x)$ is a function such that $f''(x)$ is continuous, $f'(3) > 0$, and $f''(3) < 0$, then $f(3+\Delta x) \leq f(3) + f'(3)\Delta x$ for all sufficiently small values of Δx.

True False

10. [10 points] For each of the questions below, circle all correct choices. If none of the choices are correct, circle NONE OF THESE.
You are not required to show your work on this page.

a. [2 points] Which of the following equations gives the tangent line to $y = \ln(3x+4)+1$ at $x = -1$? Circle all such equations.

i. $y = x + 2$
ii. $y = \dfrac{3}{3x+4} + 1$
iii. $y = 3x + 4$
iv. $y = 1$
v. $y = x + 4$
vi. NONE OF THESE

b. [2 points] Which of the following functions are antiderivatives of $f(x) = \cos(x)$? Circle all such functions.

i. $\frac{1}{2}(\cos(x))^2$
ii. $\sin(x) + 5$
iii. $\cos\left(x - \frac{\pi}{2}\right)$
iv. $\ln\left(3e^{\sin(x)}\right)$
v. $19 - \sin(x)$
vi. NONE OF THESE

c. [2 points] Which of the following limits equal 0? Circle all such expressions.

i. $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$
ii. $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{x}$
iii. $\lim\limits_{x\to\infty} \sin(x)$
iv. $\lim\limits_{x\to\infty} \dfrac{x^3 - 24x^2 + 188x - 480}{x^2 - 12x + 20}$
v. $\lim\limits_{x\to\infty} \dfrac{10000}{x^{1/1001}}$
vi. NONE OF THESE

d. [2 points] For K a positive constant, which of the following limits equal K? Circle all such expressions.

i. $\lim\limits_{h\to 0} \dfrac{K(1+h)^2 - K(1)^2}{h}$
ii. $\lim\limits_{h\to 0} \dfrac{K\cos(h+2\pi) - K\cos(2\pi)}{h}$
iii. $\lim\limits_{h\to 0} \dfrac{K\sin(h+2\pi) - K\sin(2\pi)}{h}$
iv. $\lim\limits_{h\to 0} \dfrac{e^{\ln(K)+h} - e^{\ln(K)}}{h}$
v. $\lim\limits_{h\to 0} \dfrac{(1+h)^K - (1)^K}{h}$
vi. NONE OF THESE

e. [2 points] For constants A and B, consider the function h defined by

$$h(t) = \begin{cases} (At)^2 - 48 & \text{if } t < 2 \\ Bt^3 & \text{if } t \geq 2. \end{cases}$$

Circle all pairs of values of A and B such that $h(t)$ is differentiable.

i. $A = 3, B = 3$
ii. $A = 6, B = 12$
iii. $A = -6, B = 12$
iv. $A = 0, B = 0$
v. $A = 18, B = 12$
vi. NONE OF THESE