5. [6 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

a. [2 points] If the function $f(x)$ is continuous on the interval $(0, 100)$, then $f(x)$ has a global maximum and a global minimum on that interval.

True $\boxed{\text{False}}$

b. [2 points] If $f(x)$ is a differentiable function with a critical point at $x = c$, then the function $g(x) = e^{f(x)}$ also has a critical point at $x = c$.

$\boxed{\text{True}}$ False

c. [2 points] If $f'(x)$ is continuous and $f'(x) \neq 0$ for all x, then $f(0) \neq f(5)$.

$\boxed{\text{True}}$ False

6. [8 points] This problem concerns the implicit curve

$$x^2 + xy + y^2 = 7$$

for which

$$\frac{dy}{dx} = \frac{-y - 2x}{x + 2y}.$$

a. [3 points] Find an equation for the tangent line to the curve at the point $(1, 2)$.

Solution:

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-2 - 2(1)}{1 + 2(2)} = -\frac{4}{5}$$

So the tangent line at $(1, 2)$ is $y = -\frac{4}{5}(x - 1) + 2$.

b. [5 points] Find the x- and y-coordinates of all points on the curve at which the tangent line is vertical.

Solution: If the tangent line is vertical, the slope will be undefined. The derivative $\frac{dy}{dx}$ is undefined when $x + 2y = 0$ which means $x = -2y$. Plugging this into the equation for the curve, we get

$$\begin{aligned}(-2y)^2 + (-2y)y + y^2 &= 7\\ y^2 &= \frac{7}{3}\\ y &= \pm\sqrt{\frac{7}{3}}\end{aligned}$$

Since $x = -2y$ this gives two points $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$ and $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$.

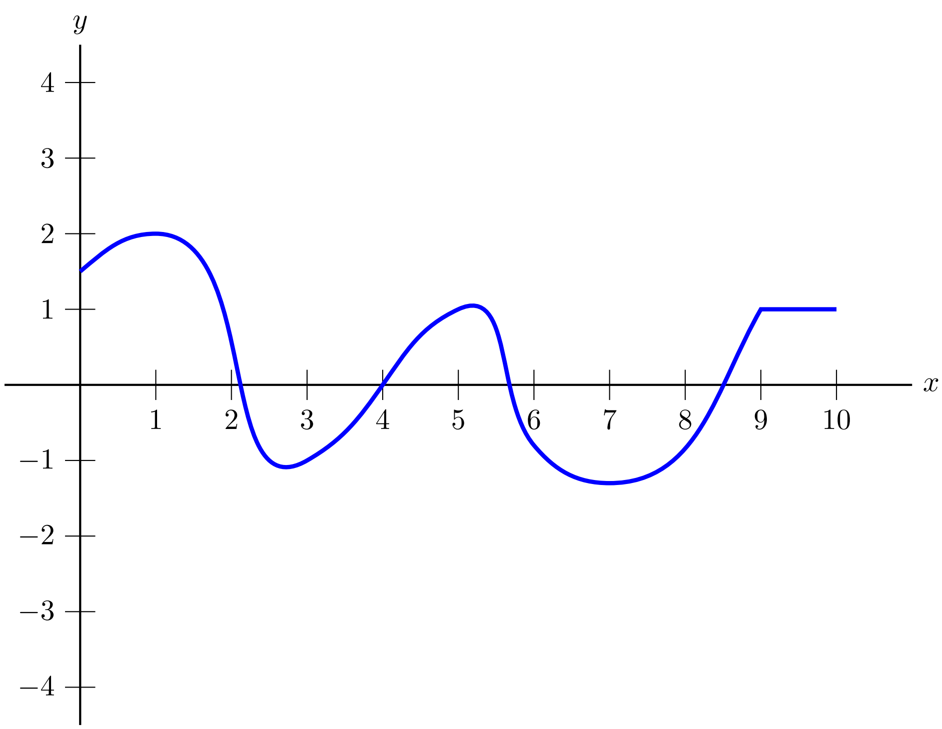
4. [10 points] A function $f(x)$ is defined and differentiable on the interval $0 < x < 10$. In addition, $f(x)$ and $f'(x)$ satisfy all of the following properties:

- $f'(x)$ is continuous on the interval $0 < x < 10$.
- $f'(1) = 2$.
- $f'(x)$ is differentiable on the interval $1 < x < 5$.
- $f(x)$ is concave up on the interval $3 < x < 5$.
- $f(x)$ has a local minimum at $x = 4$.
- $f(x)$ is decreasing on the interval $6 < x < 8$.
- $f(x)$ has an inflection point at $x = 7$.
- $f'(x)$ is not differentiable at $x = 9$.

On the axes provided below, sketch a possible graph of $f'(x)$ (the **derivative** of $f(x)$) on the interval $0 < x < 10$.
Make sure your sketch is large and unambiguous.

Solution: One possible solution is shown below.

Graph of $y = f'(x)$



7. [14 points] The table of values below gives information about the first and second derivatives of a function $f(x)$.

x	-3	-2	-1	0	1	2	3
$f'(x)$	-2	0	-1	0	2	0	-2
$f''(x)$	2	0	0	0	0	-2	-1

Assume that $f''(x)$ is **continuous** on $[-3, 3]$ and that the values of $f'(x)$ and $f''(x)$ are either **strictly positive** or **strictly negative** between consecutive table entries. You do not need to show work or give an explanation for this problem, but any unclear answers will be marked as incorrect.

a. [4 points] On which of the following intervals is $f''(x) < 0$? Circle ALL correct answers.

$-3 < x < -2$ $\boxed{-2 < x < -1}$ $-1 < x < 0$ $0 < x < 1$ $\boxed{1 < x < 2}$ $\boxed{2 < x < 3}$

b. [10 points] For each of the following x values, circle ALL answers that apply. If none of the choices apply, don't circle anything.

At $x = -2$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = -1$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = 0$, f has a **local maximum** $\boxed{\textbf{local minimum}}$ **inflection point**

At $x = 1$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = 2$, f has a $\boxed{\textbf{local maximum}}$ **local minimum** **inflection point**

1. [8 points] The table below gives several values for the function f and its derivative f'. You may assume that f is invertible and differentiable.

w	-2	-1	0	1	2
$f(w)$	1	0	-2	-3	-5
$f'(w)$	-3	-1.5	-0.5	0	-4

For each of the parts below, find the exact value of the given quantity. If there is not enough information provided to find the value, write NOT ENOUGH INFO. If the value does not exist, write DOES NOT EXIST. You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

a. [2 points] Let $h(w) = \dfrac{f(w)}{6+w}$. Find $h'(-2)$.

Solution:
$$h'(w) = \frac{f'(w)(6+w) - f(w)\cdot 1}{(6+w)^2}$$
$$h'(-2) = \frac{(-3)\cdot(4) - 1}{4^2}$$

Answer: $h'(-2) = \dfrac{-13}{16} = -0.8125$

b. [2 points] Let $k(w) = 3^{f(2w)}$. Find $k'(-1)$.

Solution: $k'(w) = \ln(3)\cdot 3^{f(2w)}\cdot f'(2w)\cdot 2$
$k'(-1) = \ln(3)\cdot 3^{f(-2)}\cdot f'(-2)\cdot 2 = \ln(3)\cdot 3^1\cdot(-3)\cdot 2$

Answer: $k'(-1) = -18\ln(3)$

c. [2 points] Let $p(w) = f(f(-w+1))$. Find $p'(1)$.

Solution: $p'(w) = f'(f(-w+1))\cdot f'(-w+1)\cdot(-1)$
$p'(1) = f'(f(0))\cdot f'(0)\cdot(-1) = f'(-2)\cdot(-0.5)\cdot(-1) = (-3)\cdot(-0.5)\cdot(-1)$

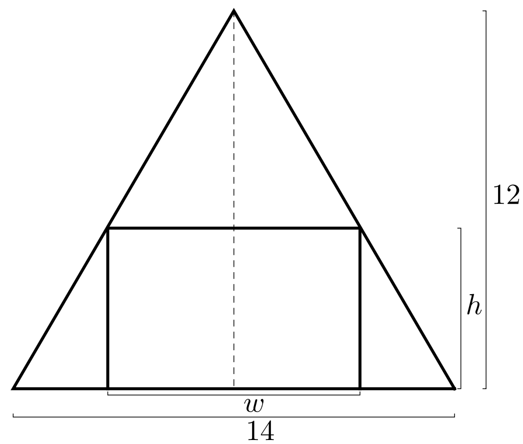
Answer: $p'(1) = -\dfrac{3}{2}$

d. [2 points] Let $r(w) = w\cdot(f(w))^2$. Find $r'(2)$.

Solution: $r'(w) = 1\cdot(f(w))^2 + w\cdot 2\cdot f(w)\cdot f'(w)$
$r'(2) = 1\cdot(f(2))^2 + 2\cdot 2\cdot f(2)\cdot f'(2) = 1\cdot(-5)^2 + 2\cdot 2\cdot(-5)\cdot(-4)$

Answer: $r'(2) = 105$

2. [5 points] Caleb has an attic apartment, and his bedroom has a triangular wall that is 14 feet wide and 12 feet tall at its tallest point. He wants to build a rectangular bookcase to put against the wall, as shown to the right. He is trying to maximize the area of the front of the bookcase.



a. [3 points] If the bookcase has width w and height h, write a formula relating w and h.

Solution: Exploiting properties of similar triangles, we get

$$\frac{w}{12-h} = \frac{14}{12},$$

so

$$w = \frac{7(12-h)}{6}.$$

b. [2 points] Using your answer from (a), find an expression for the area of the front of the bookcase in terms of the variable h.

Solution: Area $w \cdot h = \dfrac{7(12-h)h}{6}$.

3. [4 points] Suppose $g(x) = x^{2x}$. Write an explicit expression for $g'(5)$ using the limit definition of the derivative. Your expression should not contain the letter "g". Do not evaluate your expression.

Solution:

$$g'(5) = \lim_{h\to 0} \frac{g(5+h) - g(5)}{h} = \lim_{h\to 0} \frac{(5+h)^{2(5+h)} - 5^{10}}{h}.$$

7. [7 points] Alicia decides to go for a run before completing her math homework. Let $g(m)$ be the time (in hours) that Alicia spends completing her math assignment if she runs m miles. Suppose that for $1.2 \leq m \leq 8$,

$$g(m) = 2m - 12.2\ln(m) + 15 - \frac{14.4}{m}.$$

Note that on this interval, the derivative of g is given by the formula

$$g'(m) = \frac{2(m-4.5)(m-1.6)}{m^2}.$$

a. [5 points] Find all values of m that maximize and minimize the function $g(m)$ on the interval $1.2 \leq m \leq 8$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since g is continuous on the closed interval $[1.2, 8]$, by the Extreme Value Theorem g definitely attains a global maximum and global minimum on the interval, and it suffices to compare the values of $g(m)$ at the critical points and endpoints of the interval.
Notice that the critical points of $g(m)$ in the interval $1.2 \leq m \leq 8$ are at $m = 1.6$ and $m = 4.5$. Hence, we need to check the value of $g(m)$ at $m = 1.2, 1.6, 4.5, 8$:

$$g(1.2) = 2(1.2) - 12.2\ln(1.2) + 15 - \frac{14.4}{1.2} \approx 3.176$$
$$g(1.6) = 2(1.6) - 12.2\ln(1.6) + 15 - \frac{14.4}{1.6} \approx 3.466$$
$$g(4.5) = 2(4.5) - 12.2\ln(4.5) + 15 - \frac{14.4}{4.5} \approx 2.450$$
$$g(8) = 2(8) - 12.2\ln(8) + 15 - \frac{14.4}{8} \approx 3.831$$

Thus, we can see that $g(m)$ achieves its maximum on the interval at $m = 8$, and $g(m)$ achieves its minimum on the interval at $m = 4.5$.

For each answer blank below, write "NONE" if appropriate.

Answer: Global max(es) at $m=$ 8

Answer: Global min(s) at $m=$ 4.5

b. [2 points] Assuming that Alicia runs at least 1.2 miles and at most 8 miles, what is the shortest amount of time Alicia could spend completing her homework?
Remember to include units.

Solution: As we saw in part **a.**, under those assumptions, the shortest amount of time Alicia could spend completing her homework is $g(4.5) \approx 2.450$ hours.

Answer: Shortest time: 2.450 **hours**

9. [9 points] Elphaba and Walt are planning to break out of prison. They would like to escape no later than 20 hours after devising their plan, and they would like to attempt their escape during the noisiest part of the day. Let $N(t)$ be the noise level (in decibels) in the prison t hours after Elphaba and Walt have devised their escape plan.
On the interval $[0, 20]$, a formula for $N(t)$ is given by

$$N(t) = 60 + 1.01^{p(t)} \quad \text{where} \quad p(t) = \frac{1}{3}t^3 - 9t^2 + 56t + 200.$$

a. [8 points] Find the values of t that minimize and maximize $N(t)$ on the interval $[0, 20]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since $N(t)$ is continuous on the interval $[0, 20]$, we can apply the Extreme Value Theorem and compare the values of $N(t)$ at the critical points and endpoints of the interval.
We first need to find the critical points in this interval. Taking the derivative of $N(t)$ and setting it equal to zero we have $N'(t) = \ln(1.01)p'(t)(1.01)^{p(t)} = 0$ so critical points occur when $0 = p'(t) = t^2 - 18t + 56$. Solving we determine that the only critical points of $N(t)$ occur at $t = 4$ and $t = 14$, which are both in the interval $[0, 20]$.
To find the global extrema we need to evaluate the function at $t = 0, 4, 14, 20$. We find $N(0) \approx 67.316$, $N(4) \approx 80.0528$, $N(14) \approx 63.819$ and $N(20) \approx 106.874$. Choosing the largest and smallest values, by the Extreme Value Theorem, we see that the global minimum occurs at $t = 14$ and the global maximum occurs at $t = 20$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $t =$ 14

Answer: global max(es) at $t =$ 20

b. [1 point] As mentioned above, Elphaba and Walt would like to escape no later than 20 hours after devising their plan, and they would like to escape during the noisiest part of the day. When should Elphaba and Walt attempt their escape?

Answer: They should try to escape 20 hours after devising their plan.

8. [13 points] Two smokestacks d miles apart deposit soot on the ground between them. The concentration of the combined soot deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2}$$

where c and k are positive constants which depend on the quantity of smoke each stack is emitting. If $k = 27c$, find the x-value of the point on the line joining the stacks where the concentration of the deposit is a minimum. Justify that the point you found is actually a global minimum.

Solution: First we plug in $k = 27c$ and get

$$S = \frac{c}{x^2} + \frac{27c}{(d-x)^2}.$$

Then we take the derivative and set it equal to zero to find the critical points on the domain $0 < x < d$:

$$S' = \frac{-2c}{x^3} + \frac{(-2)(27c)(-1)}{(d-x)^3} = \frac{-2c}{x^3} + \frac{2\cdot 27c}{(d-x)^3} = 0$$

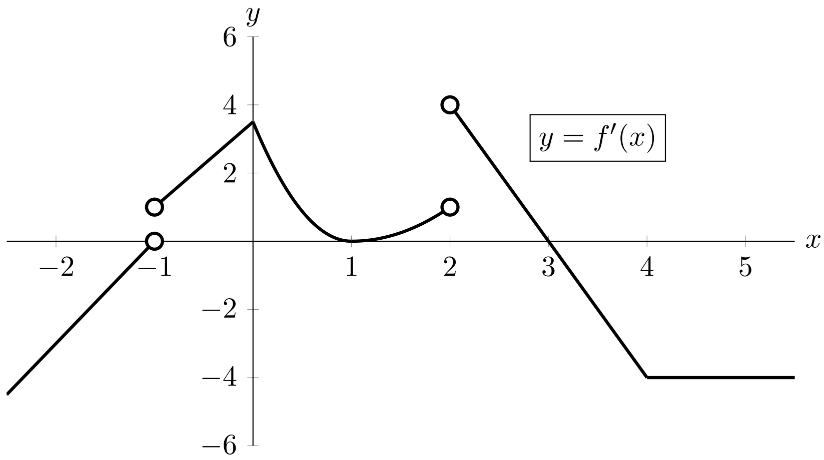
Now we solve for x:

$$\frac{2\cdot 27c}{(d-x)^3} = \frac{2c}{x^3}$$

$$\frac{27}{(d-x)^3} = \frac{1}{x^3}$$

Taking the cube root of both sides gives $\frac{3}{d-x} = \frac{1}{x}$. So $3x = d - x$. Thus $x = \frac{d}{4}$.
Now our domain is $(0, d)$ and $x = \frac{d}{4}$ is our only critical point. As $x \to 0$, $S \to \infty$, and as $x \to d$, $S \to \infty$. Thus, our global minimum must be at our sole critical point at $x = \frac{d}{4}$.

9. [12 points] The graph of a portion of the <u>derivative</u> of a function $f(x)$ is given below. Assume that the domain of f is all real numbers, and that f is continuous on the entire interval $[-2, 5]$.



Use the graph above to answer the following questions. For each question, circle <u>all</u> of the available correct answers.
(Circle NONE OF THESE if none of the available choices are correct.)

a. [2 points] At which of the following values of x does $f(x)$ appear to have a critical point?

$x = 0$ $\boxed{x = 1}$ $\boxed{x = 2}$ $\boxed{x = 3}$ $x = 4$ NONE OF THESE

b. [2 points]
At which of the following values of x does $f'(x)$ appear to have a critical point?

$\boxed{x = 0}$ $\boxed{x = 1}$ $x = 3$ $\boxed{x = 4}$ NONE OF THESE

c. [2 points] At which of the following values of x does $f(x)$ attain a local extremum?

$\boxed{x = -1}$ $x = 0$ $x = 1$ $\boxed{x = 3}$ NONE OF THESE

d. [2 points] At which of the following values of x does $f(x)$ attain a global maximum on the interval $[-1, 3]$?

$x = -1$ $x = 0$ $x = 1$ $x = 2$ $\boxed{x = 3}$ NONE OF THESE

e. [2 points] At which of the following values of x does $f(x)$ have an inflection point?

$x = -1$ $\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = 2}$ $x = 3$ NONE OF THESE

f. [2 points] For which of the following intervals is $f(x)$ concave up on the <u>entire</u> interval?

$\boxed{-1 < x < 0}$ $0 < x < 1$ $\boxed{1 < x < 2}$ $2 < x < 4$ NONE OF THESE

4. [10 points] Let $h(x)$ be a twice differentiable function defined for all real numbers x. (So h is differentiable and its derivative h' is also differentiable.)
Some values of $h'(x)$, the derivative of h are given in the table below.

x	-8	-6	-4	-2	0	2	4	6	8
$h'(x)$	3	7	0	-3	-5	-4	0	-2	6

For each of the following, circle all the correct answers.
Circle "NONE OF THESE" if none of the provided choices are correct.

a. [2 points] Circle all the intervals below in which $h(x)$ must have a critical point.

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $\boxed{2 < x < 6}$ $\boxed{6 < x < 8}$

NONE OF THESE

b. [2 points] Circle all the intervals below in which $h(x)$ must have a local extremum (i.e. a local maximum or a local minimum).

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $2 < x < 6$ $\boxed{6 < x < 8}$

NONE OF THESE

c. [2 points] Circle all the intervals below in which $h(x)$ must have an inflection point.

$\boxed{-8 < x < -4}$ $-4 < x < 0$ $0 < x < 4$ $\boxed{2 < x < 6}$ $\boxed{4 < x < 8}$

NONE OF THESE

d. [2 points] Circle all the intervals below which must contain a number c such that $h''(c) = 2$.

$\boxed{-8 < x < -6}$ $-4 < x < -2$ $-2 < x < 0$ $\boxed{2 < x < 4}$ $6 < x < 8$

NONE OF THESE

e. [2 points] Suppose that $h''(x) < 0$ for $x < -8$, and $h(-8) = 7$. Circle all the numbers below which could equal the value of $h(-10)$.

$\boxed{-2}$ $\boxed{-1}$ $\boxed{0}$ 1 2

NONE OF THESE

10. [8 points] For each question below, circle the answer that correctly completes the statement. There is exactly one correct answer per problem. You do not need to show any work or give any explanation. There is no penalty for guessing. Any unclear marks will receive no credit.

a. [2 points] Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global maximum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $x = -a$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $\boxed{x = -2a}$ $\quad$ $x = 1$

b. [2 points] Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global minimum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $\boxed{x = -a}$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $x = -2a$ $\quad$ $x = 1$

c. [2 points] If $g(x)$ is a positive differentiable function, then for $x > 0$, the derivative of the function $\dfrac{\ln x}{g(x)}$ is

$\dfrac{1}{xg(x)}$ $\quad$ $\dfrac{1}{xg'(x)}$ $\quad$ $\boxed{\dfrac{1}{xg(x)} - \dfrac{g'(x)\ln x}{(g(x))^2}}$

$\dfrac{1}{xg(x)} + \dfrac{g'(x)\ln x}{(g(x))^2}$ $\quad$ $\dfrac{1}{xg'(x)} - \dfrac{\ln x}{g(x)^2}$ $\quad$ $\dfrac{1}{xg'(x)} + \dfrac{\ln x}{g(x)^2}$

d. [2 points] Suppose the local linearization of a function $h(x)$ at the point $(x, y) = (2, -1)$ gives the estimate $h(2.1) \approx -0.88$. The value of $h'(2)$ is

0.12 $\quad$ -0.12 $\quad$ 8.8 $\quad$ -8.8 $\quad$ $\boxed{1.2}$ $\quad$ -1.2

9. [12 points] Suppose $w(x)$ is an everywhere differentiable function which satisfies the following conditions:

- $w'(0) = 0$.
- $w'(x) > 0$ for $x > 0$.
- $w'(x) < 0$ for $x < 0$.

Let $f(t) = t^2 + bt + c$ where b and c are positive constants with $b^2 > 4c$. Define $L(t) = w(f(t))$.

a. [2 points] Compute $L'(t)$. Your answer may involve w and/or w' and constants b and c.

Solution: $L'(t) = w'(t^2 + bt + c) \cdot (2t + b)$.

b. [4 points] Using your answer from (a), find the critical points of $L(t)$ in terms of the constants b and c.

Solution: $L(t)$ has critical points when $L'(t) = 0$. This happens only if $w'(t^2+bt+c) = 0$ or if $(2t + b) = 0$.
$w'(t^2 + bt + c) = 0$ means $t^2 + bt + c = 0$ by the first property of w' above. Solving using the quadratic formula, we have

$$t = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

as critical points of $L(t)$. Both of these roots exist and are distinct since $b^2 > 4c$.
If $2t + b = 0$, we have $t = -\frac{b}{2}$ as a critical point. Altogether our critical points are $t = -\frac{b}{2}, -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}, -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

c. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

Solution: For simplicity, let's set $p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ and $m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

We know that $f(t)$ is an upward opening parabola with roots at p and m. We also know $p > m$, so this means $f(t) > 0$ for $t < m$ and $t > p$. This also means $f(t) < 0$ for $m < t < p$. Thus by properties two and three of w' above we know $w'(f(t)) > 0$ for $t < m$ and $t > p$, and $w'(f(t)) < 0$ for $m < t < p$.

The expression $2t + b$ is positive for $t > -\frac{b}{2}$ and negative for $t < -\frac{b}{2}$.

Putting all of this information together gives us

$$L'(t) > 0$$

on the intervals $(m, -\frac{b}{2})$ and $(p, +\infty)$, and

$$L'(t) < 0$$

on the intervals $(-\infty, m)$ and $(-\frac{b}{2}, p)$. Thus, by the first derivative test, the critical points $t = m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$ and $t = p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ are local minima, and $t = -\frac{b}{2}$ is a local maximum.