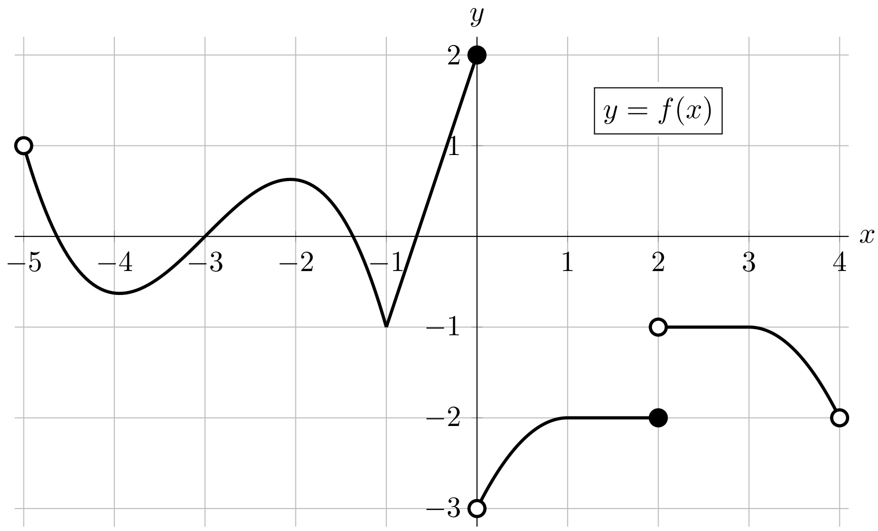
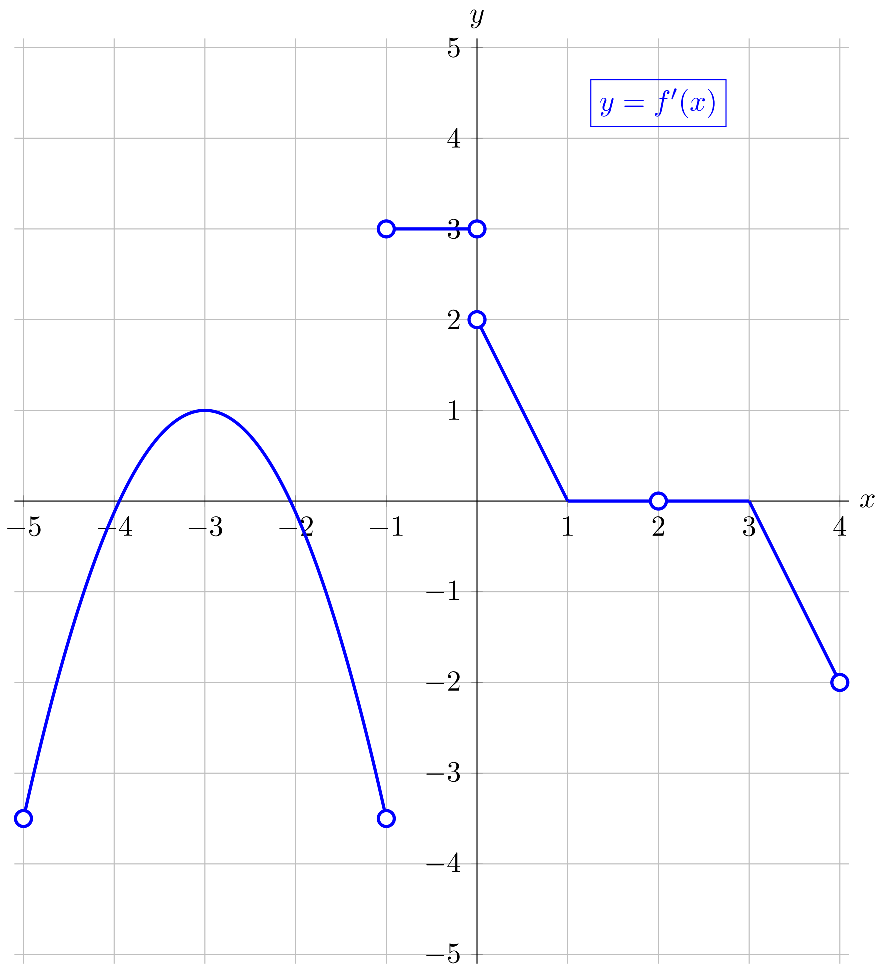
**4**. [8 points] The graph of a function $f$ is shown below.




On the axes below, sketch a graph of $f'(x)$ (the derivative of the function $f(x)$) on the interval $-5 < x < 4$. Be sure that you pay close attention to each of the following:

- where $f'$ is defined
- the value of $f'(x)$ near each of $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$
- the sign of $f'$
- where $f'$ is increasing/decreasing/constant



**2.** [14 points] Suppose $f(x)$ is a function defined for all real numbers whose derivative and second derivative are given by

$$f'(x) = (x-4)^3(x+2)^{2/3} \quad \text{and} \quad f''(x) = \frac{(11x+10)(x-4)^2}{3(x+2)^{1/3}}.$$

**a.** [7 points] Find all critical points of $f(x)$ and all values of $x$ at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

*Solution:* First we find the critical points, which occur when $f'(x) = 0$ or $f'(x)$ does not exist. From the formula, we can see that $f'(x) = 0$ when $x = 4$ or $x = -2$, and that $f'(x)$ always exists.
Next we must determine whether there is a local min, local max, or neither at each critical point. The second derivative test is inconclusive, because $f''(-2)$ does not exist and $f''(4) = 0$. So we must use the First Derivative Test. The factor $(x+2)^{2/3}$ is always positive, while $(x-4)$ is negative for $x < 4$ and positive for $x > 4$. This gives us the resulting signs:

| Interval | $x < -2$ | $-2 < x < 4$ | $4 < x$ |
|---|---|---|---|
| Sign of $f'(x)$ | $- \cdot + = -$ | $- \cdot + = -$ | $+ \cdot + = +$ |

So $f(x)$ has a local minimum at $x = 4$ and no local maxima.

**Answer:** Critical point(s) at $x =$ $-2$, $4$

Local max(es) at $x =$ NONE  Local min(s) at $x =$ $4$

**b.** [7 points] Find the $x$-coordinates of all inflection points of $f(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

*Solution:* First we find the candidate inflection points, which occur when $f''(x) = 0$ or $f''(x)$ does not exist. We can see that $f''(x) = 0$ at $x = 4$ and $-\frac{10}{11}$, and that $f''(x)$ is undefined when $x = -2$. To determine whether these are actually inflection points (where concavity changes), we must test the sign of the second derivative on either side of each of point. We find the following:

| Interval | $x < -2$ | $-2 < x < -\frac{10}{11}$ | $-\frac{10}{11} < x < 4$ | $4 < x$ |
|---|---|---|---|---|
| Sign of $f''(x)$ | $\frac{- \cdot +}{-} = +$ | $\frac{- \cdot +}{+} = -$ | $\frac{+ \cdot +}{+} = +$ | $\frac{+ \cdot +}{+} = +$ |

The inflection points of $f(x)$ occur at the points where the sign of the second derivative changes, that is, at $x = -2$ and $x = -\frac{10}{11}$.

**Answer:** Inflection point(s) at $x =$ $-2, -\dfrac{10}{11}$

**9.** [14 points]

**a.** [8 points] Consider functions $f$ satisfying <u>all</u> of the following conditions:

- $f(x)$ is differentiable on the interval $0 < x < 8$.
- The critical points of $f(x)$ in the interval $0 < x < 8$ are $x = 2, 4$, and 6. ($f(x)$ has no other critical points in this interval.)
- The table below shows some values of $f(x)$ and of its derivative $f'(x)$.

| $x$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $f(x)$ | 3 | 6 | 11 | 0 |
| $f'(x)$ | $-1$ | ? | ? | $-1$ |

For each of the statements below, decide whether the statement is true for ALL functions $f$ satisfying all of the conditions described above, for SOME of these functions $f$, or for NONE of these functions $f$. Circle the <u>one</u> correct choice for each statement.

(i) $f(x)$ has a local minimum at $x = 2$.

**[ALL]** SOME NONE

(ii) $f'(3) > 0$.

**[ALL]** SOME NONE

(iii) $f(x)$ has a local maximum at $x = 4$.

ALL **[SOME]** NONE

(iv) There is exactly one value of $a$ with $3 < a < 7$ such that $f(x)$ has a local maximum at $x = a$.

**[ALL]** SOME NONE

**b.** [6 points] Consider functions $g$ satisfying <u>all</u> of the following conditions:

- $g(z)$ and $g'(z)$ are differentiable on the interval $12 < z < 18$.
- The critical points of $g(z)$ in the interval $12 < z < 18$ are $z = 14$ and $z = 16$. ($g(z)$ has no other critical points in this interval.)
- The table below shows some values of $g(z)$ and of its <u>second derivative</u> $g''(z)$.

| $z$ | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|
| $g(z)$ | 8 | ? | 6 | ? | 2 |
| $g''(z)$ | ? | $-1$ | ? | 0 | ? |

For each of the statements below, decide whether the statement is true for ALL functions $g$ satisfying all of the conditions described above, for SOME of these functions $g$, or for NONE of these functions $g$. Circle the <u>one</u> correct choice for each statement.

(i) $g(z)$ has a local extremum at $z = 14$.

**[ALL]** SOME NONE

(ii) $g'(15) > 0$.

ALL SOME **[NONE]**

(iii) $g(z)$ has an inflection point at $z = 16$.

ALL **[SOME]** NONE

**2**. [10 points] Let $a$ be a constant with $a > 1$.
A function $w(x)$ and its derivative $w'(x)$ are given below.

$$w(x) = a + \frac{x}{x^2 + a^2} \quad \text{and} \quad w'(x) = \frac{-(x-a)(x+a)}{(x^2+a^2)^2}.$$

**a**. [5 points] Find and classify the local extrema of $w(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate.

*Solution:* By inspection of the formula for $w'$, we can see that the critical points of $w(x)$ are at $x = -a$ and $x = a$. (Note that $x^2 + a^2$ is never equal to 0 because $a \neq 0$.) The following sign chart shows the sign of $w'(x)$ on the intervals $-\infty < x < -a$, $-a < x < a$, and $a < x < \infty$. (Note that since $a$ is positive, we have $-a < a$.)

| Interval | $-\infty < x < -a$ | $-a < x < a$ | $a < x < \infty$ |
|---|---|---|---|
| sign of $w'(x)$ | $\frac{(-)(-)(-)}{+} = -$ | $\frac{(-)(-)(+)}{+} = +$ | $\frac{(-)(+)(+)}{+} = -$ |

By the First Derivative Test, we therefore see that $w(x)$ has a local minimum at $x = -a$ and a local maximum at $x = a$.

**Answer:** Local min(s) at $x =$ $-a$

**Answer:** Local max(es) at $x =$ $a$

**b**. [5 points] Find the global extrema of $w(x)$ on the interval $[1, \infty)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found the global extrema. For each answer blank, write NONE if appropriate.

*Solution:*
One solution: In part (a) we showed that $x = a$ is a local max. Since $x = a$ is the only critical point in the interval $[1, \infty)$, we conclude that $x = a$ is the global max on this interval. To determine the global min, we need to consider what happens at $x = 1$ and as $x \to \infty$. We have $\lim_{x\to\infty} w(x) = a$. We also have $w(1) = a + \frac{1}{1+a^2}$, which is larger than $a$, so $x = 1$ is not the global min. Since $w(x)$ decreases to $a$ (but never quite reaches it) as $x \to \infty$, we conclude that $w(x)$ has no global min on the interval $[1, \infty)$.

Another solution: We check the values of $w(x)$ at $x = 1$ and $x = a$ (note that since $a > 1$, $a$ is always in the interval $[1, \infty)$, and $-a$ is never in this interval) and then consider what happens as $x$ goes to infinity.
We have $w(1) = a + \frac{1}{1+a^2}$ and $w(a) = a + \frac{1}{2a}$. Thus we need to compare $\frac{1}{1+a^2}$ and $\frac{1}{2a}$. These are equal if $a = 1$ (but we know $a > 1$), and otherwise, $\frac{1}{2a}$ is larger. Since $w(x)$ is decreasing for $x > a$, the global max is $\frac{1}{2a}$.
We have $\lim_{x\to\infty} w(x) = a$ but $w(x)$ is never equal to $a$. Since this limit is smaller than $w(1)$ and $w(a)$, we conclude that $w(x)$ has no global minimum on the interval $[1, \infty)$.

**Answer:** Global min(s) at $x =$ NONE

**Answer:** Global max(es) at $x =$ $a$

**8**. [11 points] A function $g(x)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \quad \text{and} \quad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

**a**. [2 points] Find the $t$-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact $t$-coordinates.

*Solution:* Since $g'(t)$ is defined for all $t$, the only critical points occur where $g'(t) = 0$. To find these $t$ values, we use the Quadratic Formula:

$$t = 3 \pm \sqrt{9-6} = 3 \pm \sqrt{3}.$$

**Answer:** Critical point(s) at $t =$ $3 - \sqrt{3}$, $3 + \sqrt{3}$

**b**. [6 points] For each of the following, find the values of $t$ that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of $t$ that maximize and minimize $g(t)$ on the interval $[0, 8]$.

*Solution:* Since $[0, 8]$ is a closed interval, by the Extreme Value Theorem, $g(t)$ must have both a global max and a global min, occuring either at a critical point or at an endpoint. We therefore make a table of values at $t = 0, t = 3 - \sqrt{3} \approx 1.27$, $t = 3 + \sqrt{3} \approx 4.73$, and $t = 8$:

| $t$ | 0 | 1.27 | 4.73 | 8 |
|---|---|---|---|---|
| $g(t)$ | 20 | 5.69 | 14.01 | 9.16 |

From the table, we see the global max at $t = 0$ and the global min at $t = 3 - \sqrt{3} \approx 1.27$.

**Answer:** Global max(es) at $t =$ 0 Global min(s) at $t =$ $3 - \sqrt{3}$

(ii) Find the values of $t$ that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

*Solution:* Note that only one of our critical points, $t = 3 + \sqrt{3} \approx 4.73$, lies in this interval. Global extrema, if they exist, then, can only occur at $t = 4$ and $t = 3 + \sqrt{3}$, so we make a table of these values:

| $t$ | 4 | 4.73 |
|---|---|---|
| $g(t)$ | 13.53 | 14.01 |

We must also consider the behavior as $t \to \infty$, the open endpoint of our interval. Note that $g'(t) < 0$ for $t > 3 + \sqrt{3}$, so $g(t)$ is decreasing for $t > 3 + \sqrt{3}$. So the global max occurs at the largest value in the table, at $t = 3 + \sqrt{3} \approx 4.73$. As $t$ gets larger and larger, $g(t) = \frac{10(t^2-2t+2)}{e^{0.5t}}$ tends to 0, as $e^{0.5t}$ grows faster than any polynomial in the long run. Since this limiting value of 0 is smaller than every value in our table, there is no global min.

**Answer:** Global max(es) at $t =$ $3 + \sqrt{3}$ Global min(s) at $t =$ NONE

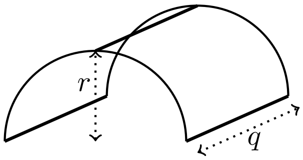
**c**. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the $t$-coordinates of all critical points and inflection points of $G(t)$. For each answer black, write NONE if appropriate. You do not need to justify your answers.

*Solution:* Critical points of $G(t)$ are zeros of $g(t)$, of which there are none. Inflection points of $G(t)$ are local extrema of $g(t)$, which occur at $t = 3 \pm \sqrt{3}$ (which we know to be local extrema because they are in fact global extrema in the interiors of some intervals).

**Answer:** Critical point(s) at $t =$ NONE

**Answer:** Inflection point(s) at $t =$ $3 - \sqrt{3}$, $3 + \sqrt{3}$

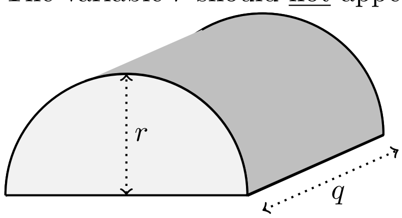
**3**. [9 points] Duncan's person is making him a new tent in the shape of half a cylinder. She plans to use wire to make the tent frame. This will consist of two semicircles of radius $r$ (measured in inches) attached to three pieces of wire of length $q$ (also measured in inches), as shown in the diagram below. She has 72 inches of wire to use for this.




**a**. [4 points] Find a formula for $r$ in terms of $q$.

*Solution:* The amount of wire $S$ used on the one semicircle of radius $r$ is given by $S = \frac{1}{2}(2\pi r)$ inches. For the rest of the tent, she uses $3q$ inches of wire. Since she used 72 inches of wire to build the tent, we have $r = \dfrac{72-3q}{2\pi}$.

**b**. [2 points] Let $V(q)$ be the volume (in cubic inches) of the space inside the tent after the fabric is added, given that the total length of wire is 72 inches and the length of the tent is $q$ inches. (Recall that the tent shape is half of a cylinder.) Find a formula for $V(q)$. The variable $r$ should not appear in your answer.




*Solution:* The volume of enclosed by the tent $V$ is the volume of a half cylinder. In this case $V = \frac{1}{2}\pi r^2 h$, where $r$ is the radius of the semicircular lateral face and $h$ is the length of the tent. In our case $h = q$ and using our answer from part **a** we obtain

$$V = \frac{1}{2}\pi r^2 h = \frac{\pi q}{2}\left(\frac{72-3q}{2\pi}\right)^2$$

**c**. [3 points] In the context of this problem, what is the domain of $V(q)$?

*Solution:* The length $q$ of the tent cannot be negative and it has to be smaller than the total amount of wire used 72 inches, then $0 < q < 72$. On the other hand, the more wire you use building the length of the tent $q$ the smallest the radius $r$ will be. If we set our expression for $r$ in terms of $q$ from **a.** to 0: $\dfrac{72-3q}{2\pi} = 0$, we obtain $3q = 72$ and therefore $q = 24$. Hence the domain of $V(q)$ is all values of $q$ that satisfy $0 < q < 24$.

**4.** [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that $t$ seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where $k$ and $c$ are nonzero constants.

**a.** [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of $t$. The constants $k$ and $c$ may appear in your answer.

*Solution:* The velocity is the derivative of the height function, so we compute

$$v(t) = h'(t) = -15k\sin(kt)\,.$$

Notice that the Chain Rule gives us a factor of $k$ out front, and since $c$ is an additive constant, it disappears when we take the derivative.
Notice also that $v(t) = \frac{dh}{dt}$ does indeed have units of feet per second, as required.

**Answer:** $v(t) =$ $-15k\sin(kt)$

**b.** [2 points] Find a formula for $v'(t)$. The constants $k$ and $c$ may appear in your answer.

**Answer:** $v'(t) =$ $-15k^2\cos(kt)$

**c.** [3 points] What is the maximum vertical acceleration experienced by the captain? The constants $k$ and $c$ may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

*Solution:* The acceleration is just the derivative of the velocity function, which was just computed in the previous part.
Since $v'(t) = -15k^2\cos(kt)$ is sinusoidal with midline 0 and amplitude $15k^2$, the maximum value it achieves is $15k^2$.
Since $v'(t) = \frac{dv}{dt}$, the units on the acceleration are feet per second per second, or feet per second squared.

**Answer:** Max vertical acceleration: $15k^2$ **ft/s$^2$**

**9.** [10 points] Our friend Oren, the Math 115 student, wants to minimize how long it will take him to complete his upcoming web homework assignment. Before starting the assignment, he buys a cup of tea containing 55 milligrams of caffeine.
Let $H(x)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes $x$ milligrams of caffeine. For $10 \le x \le 55$

$$H(x) = \frac{1}{120}x^2 - \frac{4}{3}x + 20\ln(x).$$

Instead of immediately starting the assignment, he solves a calculus problem to determine how much caffeine he should consume.

**a.** [8 points] Find all the values of $x$ at which $H(x)$ attains global extrema on the interval $10 \le x \le 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

*Solution:* Since $H(x)$ is continuous on the interval $10 \le x \le 55$, by the Extreme Value Theorem, $H(x)$ attains both a global minimum and a global maximum on this interval. These will occur at either endpoints or critical points.
Now,

$$H'(x) = \frac{x}{60} - \frac{4}{3} + \frac{20}{x} = \frac{x^2 - 80x + 1200}{60x} = \frac{(x-60)(x-20)}{60x}.$$

Thus, $H(x)$ has exactly one critical point on the interval $10 \le x \le 55$, and it is at $x = 20$. To determine the global extrema, we compare the values of $H(x)$ at all critical points and endpoints

| $x$ | 10 | 20 | 55 |
|---|---|---|---|
| $H(x)$ | $\approx 33.55$ | $\approx 36.58$ | $\approx 32.02$ |

Thus, the global minimum is at $x = 55$, and the global maximum is at $x = 20$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

**Answer:** global min(s) at $x =$ 55

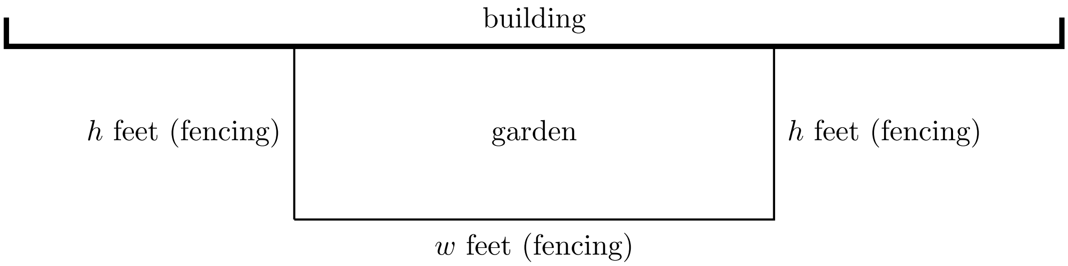
**Answer:** global max(es) at $x =$ 20

**b.** [2 points] Assuming Oren consumes at least 10 milligrams and at most 55 milligrams of caffeine, what is the shortest amount of time it could take for him to finish his assignment? *Remember to include units.*

*Solution:* The minimum of $H(x)$ occurs at $x = 55$, where $H(55) \approx 32.02$.

**Answer:** $\approx 32$ **minutes**

**4.** [12 points] Researchers are constructing a rectangular garden adjacent to their building. The garden will be bounded by the building on one side and by a fence on the other three sides. (See diagram below.) The fencing will cost them \$5 per linear foot. In addition, they will also need topsoil to cover the entire area of the garden. The topsoil will cost \$4 per square foot of the garden's area.
Assume the building is wider than any garden the researchers could afford to build.




**a.** [5 points] Suppose the garden is $w$ feet wide and extends $h$ feet from the building, as shown in the diagram above. Assume it costs the researchers a total of \$250 for the fencing and topsoil to construct this garden. Find a formula for $w$ in terms of $h$.

*Solution:* A garden of these dimensions will require $2h+w$ feet of fencing and $hw$ square feet of ground covered by topsoil. Thus,

$$250 = 5(2h + w) + 4hw\,.$$

Solving for $w$ we find

$$w = \frac{250 - 10h}{4h + 5}\,.$$

**Answer:** $w = \frac{250 - 10h}{4h + 5}$

**b.** [3 points] Let $A(h)$ be the total area (in square feet) of the garden if it costs \$250 and extends $h$ feet from the building, as shown above. Find a formula for the function $A(h)$. The variable $w$ should not appear in your answer.

(Note that $A(h)$ is the function one would use to find the value of $h$ maximizing the area. You should not do the optimization in this case.)

*Solution:* The area of the garden in square feet is given by $hw$. In part (a), a formula for $w$ in terms of $h$ was found when $h$ and $w$ are the dimensions of a garden that will cost \$250 in supplies to construct. Thus, $A(h) = h\left(\frac{250 - 10h}{4h + 5}\right)$.

**Answer:** $A(h) = h\left(\frac{250 - 10h}{4h + 5}\right)$

**c.** [4 points] In the context of this problem, what is the domain of $A(h)$?

**Answer:** $0 < h < 25$

**4.** [11 points] Elphaba has found a corrupt prison guard, Mert, to sell her metal piping to use to dig a tunnel out of the prison. Mert can sell Elphaba steel piping and copper piping, and he provides the following information.

- The number of kilograms (kg) of soil that Elphaba can dig with steel piping is proportional to the number of centimeters (cm) of steel piping that she buys. She can dig 50 kg of soil per cm of steel piping, and her cost (in dollars) of buying $x$ cm of steel piping is given by $A(x) = x^2 + x$.
- The number of kilograms (kg) of soil that Elphaba can dig with copper piping is proportional to the number of centimeters (cm) of copper piping that she buys. She can dig 30 kg of soil per cm of copper piping, and her cost (in dollars) of buying $y$ cm of copper piping is given by $B(y) = 2y$.

**a.** [1 point] How many kilograms of soil can Elphaba dig with $x$ cm of steel piping?

**Answer:** $50x$

For parts (b)-(d) below, suppose Elphaba buys $w$ cm of steel piping and $k$ cm of copper piping and that this is exactly the right amount of piping so that she can dig through 2700 kg of soil to dig her escape tunnel.

**b.** [3 points] Write a formula for $k$ in terms of $w$.

*Solution:* With $w$ cm of steel piping, she can dig through $50w$ kg of soil and with $k$ cm of copper piping, she can dig through $30k$ kg of soil. So $50w + 30k = 2700$, and solving for $k$, we find $k = \frac{2700-50w}{30}$.

**Answer:** $k = \dfrac{2700-50w}{30} = 90 - \dfrac{5}{3}w$

**c.** [4 points] Let $T(w)$ be the total cost (in dollars) of all the piping Elphaba buys to dig her escape tunnel. Find a formula for the function $T(w)$. The variable $k$ and the function names $A$ and $B$ should not appear in your answer.
(Note that $T(w)$ is the function one would use to minimize Elphaba's costs. You should not do the optimization in this case.)

*Solution:* $$T(w) = A(w) + B\left(\frac{2700-50w}{30}\right) = w^2 + w + 2\left(\frac{2700-50w}{30}\right).$$

**Answer:** $T(w) = w^2 + w + 2\left(\dfrac{2700-50w}{30}\right)$ **or** $w^2 - \dfrac{7}{3}w + 18$

**d.** [3 points] What is the domain of $T(w)$ in the context of this problem?

*Solution:* In the context of this problem, the smallest possible value of $w$ is 0, which would occur if Elphaba were to buy only copper piping. The largest possible value of $w$ would occur if Elphaba were to buy only steel piping. In that case, $50w = 2700$ so $w = \frac{2700}{50} = 54$. In the context of this problem, the domain of $T(w)$ consists of all values of $w$ between 0 and 54, i.e. the interval $[0, 54]$.

**Answer:** $[0, 54]$