5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At $(1, 1)$, the right hand side is $\sin(\pi) = 0$ and the left hand side is $1 - 1 = 0$. Therefore the point is on the curve since the right and left hand sides are equal.

b. [5 points] Calculate $\frac{dy}{dx}$.

Solution: Taking the derivative with respect to x of the equation, we have

$$\pi\cos(\pi xy) \cdot (y + x\frac{dy}{dx}) = \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{\pi y\cos(\pi xy)}{1 - \pi x\cos(\pi xy)}.$$

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution: The slope of the tangent line to the curve is

$$\frac{dy}{dx}(1, 1) = \frac{\pi\cos(\pi)}{1 - \pi\cos(\pi)} = \frac{-\pi}{1 + \pi}.$$

The equation for the tangent line is

$$y - 1 = \frac{-\pi}{1 + \pi}(x - 1).$$

3. [12 points] The following questions relate to the implicit function

$$y^2 + 4x = 4xy^2.$$

a. [4 points] Compute $\dfrac{dy}{dx}$.

Solution: Differentiating the equation with respect to x, we have

$$2y\frac{dy}{dx} + 4 = 4y^2 + 8xy\frac{dy}{dx}.$$

Gathering terms involving $\frac{dy}{dx}$ to one side, the equation becomes

$$2y\frac{dy}{dx} - 8xy\frac{dy}{dx} = 4y^2 - 4$$

which gives the solution

$$\frac{dy}{dx} = \frac{4y^2 - 4}{2y - 8xy}.$$

b. [4 points] Find the equation for the tangent line to this curve at the point $(\frac{1}{3}, 2)$.

Solution: The slope is

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{3},2)} = \frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = -9,$$

so by the point-slope formula, the equation is

$$y = -9x + 5.$$

c. [4 points] Find the x- and y-coordinates of all points at which the tangent line to this curve is vertical.

Solution: The slope is undefined as these points, so we must have $2y - 8xy = 0$. Factoring out a $2y$ we get

$$2y(1 - 4x) = 0$$

which gives the solutions $y = 0$ or $x = \frac{1}{4}$. Plugging into the equation for the implicit function, $y = 0$ gives the point $(0, 0)$. However, when we plug in $x = \frac{1}{4}$, we get the equation $y^2 + 1 = y^2$, which has no solutions. Therefore, $(0, 0)$ is the only point at which the tangent line is vertical.

11. [5 points] A curve $\mathcal{C}$ gives y as an implicit function of x. The curve $\mathcal{C}$ passes through the point $(1, 2)$ and satisfies

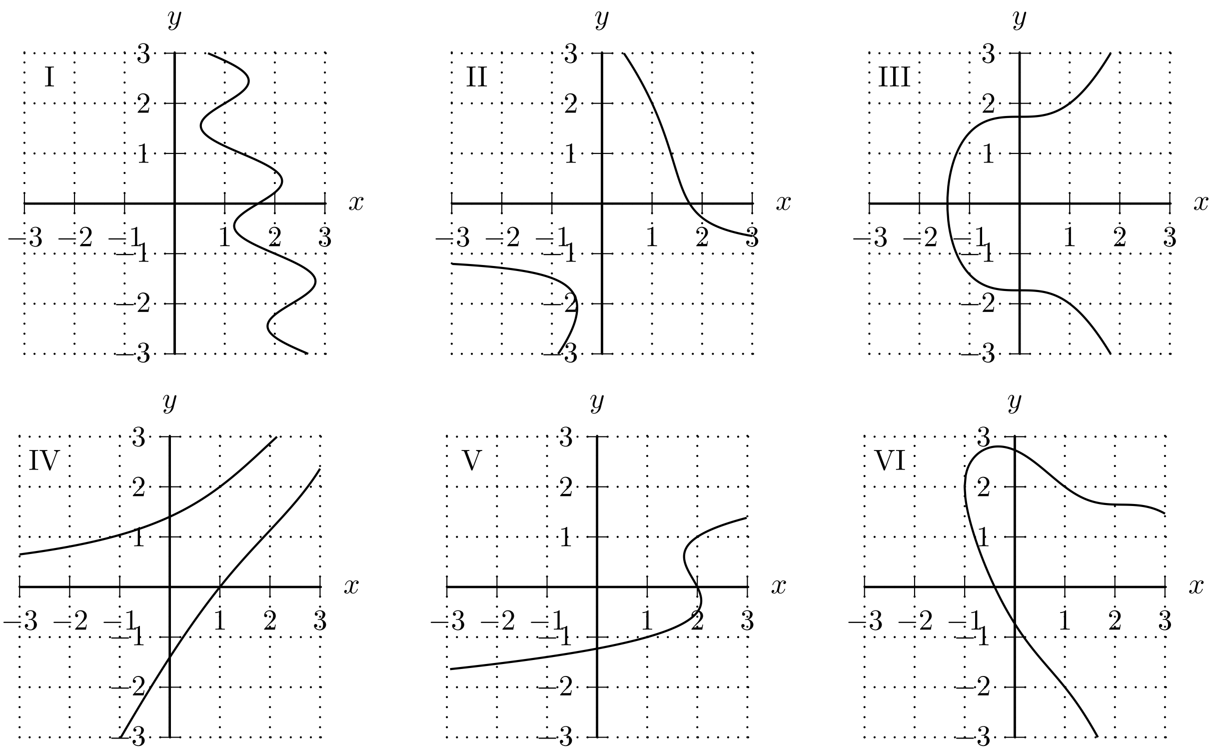
$$\frac{dy}{dx} = \frac{y^2 - 2xy + 4y - 5}{4(y - x)}.$$

a. [1 point] One of the values below is the slope of the curve $\mathcal{C}$ at the point $(1, 2)$. Circle that one value.

Solution: Plugging $x = 1$ and $y = 2$ into the given formula for $\frac{dy}{dx}$ yields $3/4$.

Answer: The slope at $(1, 2)$ is $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{5}{8}$ $\frac{2}{3}$ $\boxed{\frac{3}{4}}$ $\frac{4}{5}$

b. [4 points] One of the following graphs is the graph of the curve $\mathcal{C}$.
Which of the graphs I-VI is it? To receive any credit on this question, you <u>must</u> circle your answer next to the word "Answer" below.



Solution: We know that the desired curve passes through the point $(1, 2)$ with slope $3/4$. This allows us to eliminate Graph V (which doesn't pass through $(1, 2)$) and Graphs II and VI (which have negative slope at $(1, 2)$).
To decide between Graphs I, III, and IV, we look at other points on the graphs.
Graph I passes through the point $(2, -1)$ with negative slope, but the above formula for $\frac{dy}{dx}$ says that it should have positive slope there, so Graph I is incorrect.
Graph III passes through the point $(1, -2)$ with negative slope, but the above formula for $\frac{dy}{dx}$ says that it too should have positive slope there, so Graph III is incorrect.
The only remaining possibility is Graph IV.
(Note that we could have also eliminated all but Graph IV by checking for vertical tangent lines at points (x, y) with $y = x$.)

Remember: To receive any credit on this question, you <u>must</u> circle your answer next to the word "Answer" below.

Answer: I II III $\boxed{\text{IV}}$ V VI

7. [6 points] A curve $\mathcal{C}$ gives y as an implicit function of x. This curve passes through the point $(-2, 1)$ and satisfies

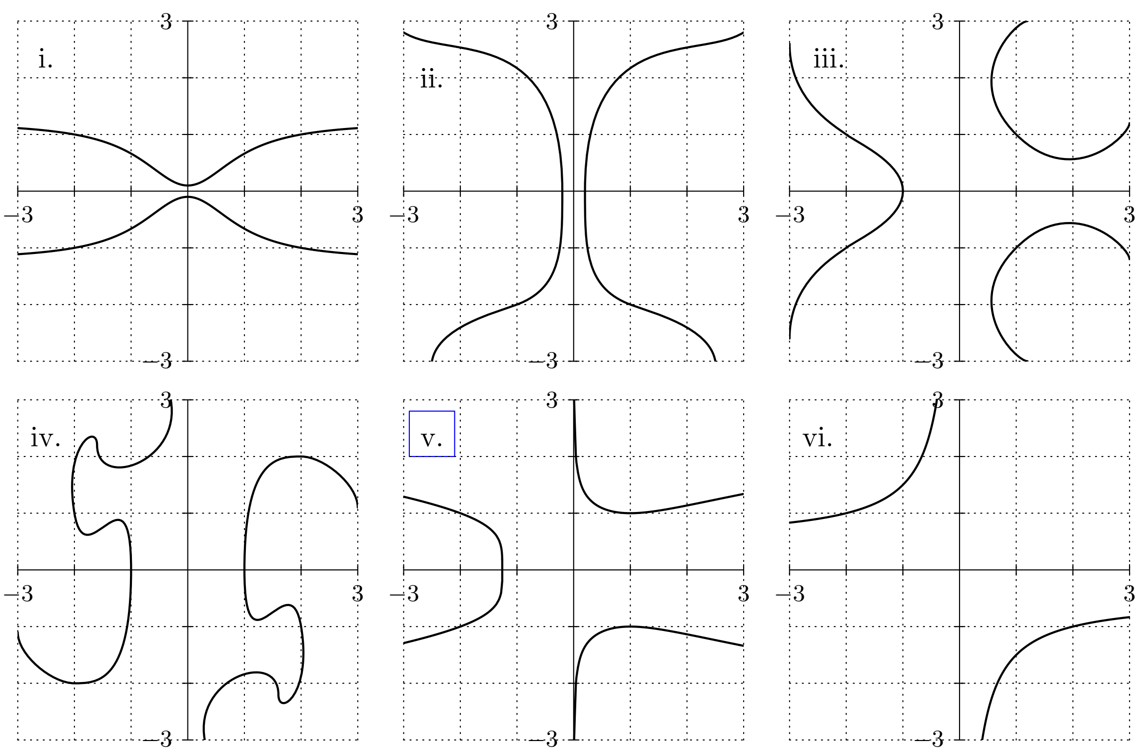
$$\frac{dy}{dx} = \frac{x^2 - y^4}{2xy^3}.$$

a. [1 point] One of the values below is the slope of the curve $\mathcal{C}$ at the point $(-2, 1)$. Circle that one value.

Answer: The slope at $(-2, 1)$ is

$-\frac{3}{16}$ $-\frac{1}{4}$ $-\frac{3}{8}$ $-\frac{1}{2}$ $-\frac{5}{8}$ $\boxed{-\frac{3}{4}}$ $-\frac{15}{16}$

b. [5 points] One of the following graphs is the graph of the curve $\mathcal{C}$. Which of the graphs i-vi is it? To receive any credit on this question, you <u>must</u> circle your answer next to the word "Answer" below.

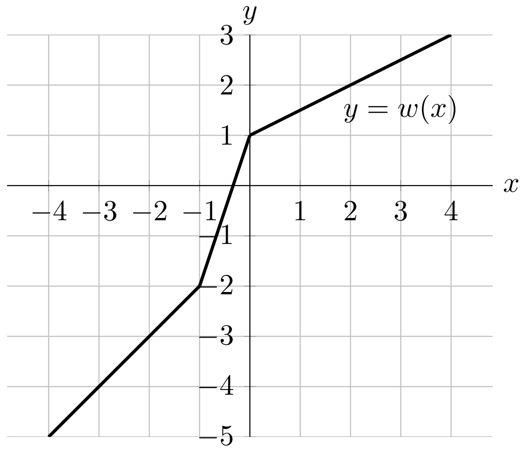


Remember: To receive any credit on this question, you <u>must</u> circle your answer next to the word "Answer" below.

Answer: i. ii. iii. iv. $\boxed{\text{v.}}$ vi.

Solution: The curve must pass through the point $(-2, 1)$, which rules out (ii). As seen in part (a), the slope at $(-2, 1)$ is negative, which rules out (vi). The tangent lines must be horizontal when the curve crosses the x- or y-axis, which rules out (i). Graph (iv) can be ruled out in a number of ways: the magnitude of the slope is too large at $(-2, 1)$, there should not be vertical tangent lines away from the axes, and there should not be a horizontal tangent line at $(2, 2)$. Finally, there should be a horizontal tangent through $(1, 1)$, ruling out (iii). This leaves graph (v).
Note: The slope at the point $(-2, 1)$ in graph (v) as it appears here is not sufficiently steep. For this reason, full credit was also awarded for choosing graph (iii).

4. [10 points] A portion of the graph of the function $w(x)$ is shown below.



For each of the parts below, find the value of the given quantity. If there is not enough information provided to find the value, write NOT ENOUGH INFO. If the value does not exist, write DOES NOT EXIST. You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown. All your answers must be in **exact** form.

a. [2 points] Let $k(x) = w^{-1}(x)$. Find $k'(-1.5)$.

Solution: $k'(x) = \dfrac{1}{w'(w^{-1}(x))}$ so $k'(-1.5) = \dfrac{1}{w'(w^{-1}(-1.5))} = \dfrac{1}{w'(-5/6)} = \dfrac{1}{3}$

Answer: $k'(-1.5) =$ $\dfrac{1}{3}$

b. [2 points] Let $h(u) = \ln(3w(u))$. Find $h'(1)$.

Solution: $h'(u) = \dfrac{1}{3w(u)}3w'(u)$ so $h'(1) = \dfrac{1}{3w(1)} \cdot 3w'(1) = \dfrac{w'(1)}{w(1)} = \dfrac{1/2}{3/2} = \dfrac{1}{3}$.

Answer: $h'(1) =$ $\dfrac{1}{3}$

c. [2 points] Let $n(x) = \dfrac{w(x)}{1-x^2}$. Find $n'(-2)$.

Solution: $n'(x) = \dfrac{w'(x)(1-x^2) - w(x)(-2x)}{(1-x^2)^2}$ so

$$n'(-2) = \frac{w'(-2)(1-(-2)^2) - w(-2)(-2\cdot -2)}{(1-(-2)^2)^2} = \frac{(1)(-3) - (-3)(4)}{3^2} = \frac{9}{9}$$

Answer: $n'(-2) =$ 1

d. [2 points] Let $s(x)$ be the exponential function $s(x) = 4^{w(x)}$. Find $s'(2)$.

Solution: $s'(x) = \ln(4)\cdot 4^{w(x)}w'(x)$ so $s'(2) = \ln(4)\cdot 4^{w(2)}\cdot w'(2) = \ln(4)\cdot 4^2 \cdot \frac{1}{2}$

Answer: $s'(2) =$ $8\ln(4)$

e. [2 points] Let $p(x) = x \cdot w^{-1}(x)$. Find $p'(-1)$.

Solution: $p'(x) = 1\cdot w^{-1}(x) + x \cdot \dfrac{1}{w'(w^{-1}(x))}$ so $p'(-1) = -\dfrac{2}{3} + -1 \cdot \dfrac{1}{3} = -1$

Answer: $p'(-1) =$ -1

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}(\cos(ax) + by\ln(x)) &= \frac{d}{dx}(3+y^3) \\ -a\sin(ax) + \frac{by}{x} + b\ln(x)\frac{dy}{dx} &= 3y^2\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)} \end{aligned}$$

Answer: $\frac{dy}{dx} = \boxed{\frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)}}$

11. [6 points] Let $h(x) = x^x$. For this problem, it may be helpful to know the following formulas:

$$h'(x) = x^x(\ln(x)+1) \qquad \text{and} \qquad h''(x) = x^x\left(\frac{1}{x} + (\ln(x)+1)^2\right).$$

a. [2 points] Write a formula for $p(x)$, the local linearization of $h(x)$ near $x = 1$.

Solution: $h(1) = 1$ and $h'(1) = 1^1(\ln(1)+1) = 1$, so $p(x) = 1 + 1\cdot(x-1) = x$.

Answer: $p(x) =$ x

b. [4 points] Write a formula for $u(x)$, the quadratic approximation of $h(x)$ at $x = 1$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Solution: $h''(1) = 1(1 + (0+1)^2) = 2$, so $u(x) = 1 + (x-1) + \frac{2}{2}(x-1)^2 = x^2 - x - 1$.

Answer: $u(x) =$ $1 + (x-1) + (x-1)^2\ (= x^2 - x - 1)$

4. [13 points] Let $f(x) = e^{\sin\sqrt{x}}$. Let P be the point on the graph of f at which $x = 4\pi^2 (\approx 39.4784)$.

a. [3 points] Calculate $f'(x)$.

Solution:

$$f'(x) = \left(e^{\sin\sqrt{x}}\right)\left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$$

b. [4 points] Find an **exact** formula for the tangent line $L(x)$ to $f(x)$ at P. **Exact** means your answer should not involve any decimal approximations.

Solution:

$$\text{slope} = f'(4\pi^2) = \frac{e^{\sin(2\pi)}\cos(2\pi)}{2\cdot 2\pi} = \frac{1}{4\pi},$$

so $L(x) = \frac{x}{4\pi} + b$, where b is the vertical intercept. When $f(4\pi^2) = e^{sin(2\pi)} = 1$, so $1 = \frac{4\pi^2}{4\pi} + b$, which gives us $b = 1 - \pi$, so

$$L(x) = \frac{x}{4\pi} + 1 - \pi$$

.

c. [2 points] Use your formula for $L(x)$ to approximate $e^{\sin\sqrt{38}}$.

Solution:

$$e^{\sin\sqrt{38}} = f(38) \approx L(38) = \frac{38}{4\pi} + 1 - \pi \approx 0.8824.$$

d. [4 points] Recall that the error, $E(x)$, is the actual value of the function minus the value approximated by the tangent line. Given the fact that in this case $E(39) \approx 0.000613$ and $E(40) \approx 0.000719$, would you expect $f''(4\pi^2)$ to be positive or negative? Explain, without doing any calculations.

Solution: The errors are positive, which means that near P the tangent line lies below the curve, so the function is probably concave up at P. Since concave up corresponds to positive second derivative, we should expect the sign of $f''(4\pi^2)$ to be positive.

6. [9 points] A group of biology students is studying the length L of a newborn corn snake (in cm) as a function of its weight w (in grams). That is, $L = G(w)$. A table of values of $G(w)$ is shown below.

w	5	10	15	20	25
$G(w)$	24.5	31.6	38.7	44.7	50
$G'(w)$	2.23	1.58	1.30	1.12	1.05

Assume that $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$.

a. [2 points] Find a formula for $H(w)$, the tangent line approximation of $G(w)$ near $w = 20$.

Solution: The formula for is $H(w) = G(20) + G'(20)(w - 20)$. From the table we get $H(w) = 44.7 + 1.12(w - 20)$.

b. [1 point] Use the tangent line approximation of $G(w)$ near $w = 20$ to approximate the length of a corn snake that weighs 22 grams.

Solution: $G(22) \approx H(22) = 1.12(22 - 20) + 44.7 = 46.94$ cm.

c. [2 points] Is your answer in part (b) an overestimate or an underestimate? Circle your answer and write a sentence to justify it.

Solution:

Circle one: $\boxed{\text{Overestimate}}$ Underestimate CANNOT BE DETERMINED

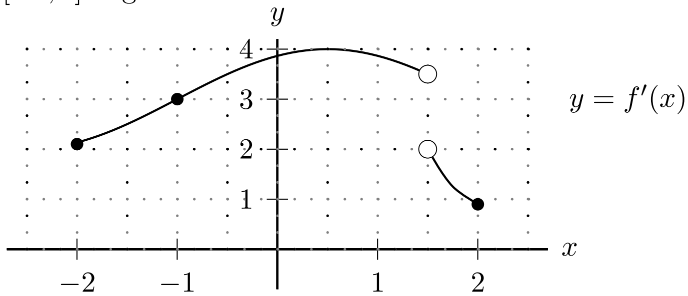
Justification:
Since $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$, then $G(w)$ is concave down on $0 < w < 25$. Hence the values of the tangent line approximation $H(w)$ will be larger than the actual values of $G(w)$ for $0 < w < 25$.

d. [4 points] In their study of the growth of corn snakes, they found the results of a recent article that states that the average weight w of a corn snake (in grams) t weeks after being born is given by $w = \frac{1}{5}t^2$. Let $S(t) = G\left(\frac{1}{5}t^2\right)$ be the length of a corn snake t weeks after being born. Find a formula for $P(t)$, the tangent line approximation of $S(t)$ near $t = 5$.

Solution: The formula for the tangent line approximation $P(t)$ is $P(t) = S(5) + S'(5)(t - 5)$. Since $S(t) = G(\frac{1}{5}t^2)$, then $S'(t) = \frac{2}{5}t \cdot G'(\frac{1}{5}t^2)$. Using these formulas we get that $S(5) = G(\frac{1}{5}(5^2)) = G(5) = 25.4$ and $S'(5) = 2 \cdot G'(5) = 4.46$.

Answer: $P(t) =$ $\underline{24.5 + 4.46(t - 5) = 4.46t + 2.2}$

3. [8 points] Suppose $f(x)$ is a function that is continuous on the interval $[-2, 2]$. The graph of $f'(x)$ on the interval $[-2, 2]$ is given below.



a. [3 points] Let $L(x)$ be the local linearization of $f(x)$ at $x = -1$. Using the fact that $f(-1) = 4$, write a formula for $L(x)$.

Solution: $f(-1) = 4$ and $f'(-1) = 3$, so $L(x) = 4 + 3(x - (-1)) = 4 + 3(x + 1)$.

Answer: $L(x) =$ $4 + 3(x + 1)$ **or** $3x + 7$

b. [2 points] Use your formula for $L(x)$ to approximate $f(-0.5)$.

Solution: Since -0.5 is close to -1 we have

$$f(-0.5) \approx L(-0.5) = 4 + 3(-0.5 + 1) = 43(0.5) = 5.5.$$

Answer: $f(-0.5) \approx$ 5.5

c. [3 points] Is your answer from part (b) an overestimate or an underestimate of the actual value of $f(-0.5)$? Justify your answer.

Circle one: overestimate $\boxed{\text{underestimate}}$ CANNOT BE DETERMINED

Justification:

Solution: The function $f'(x)$ is increasing between -2 and 0 so $f(x)$ is concave up over this interval. Therefore the tangent line to the graph of $f(x)$ at $x = -1$ lies below the graph of $f(x)$ between $x = -2$ and $x = 0$. In particular, the local linearization $L(x)$ of $f(x)$ at $x = -1$ gives an underestimate of f on that interval.