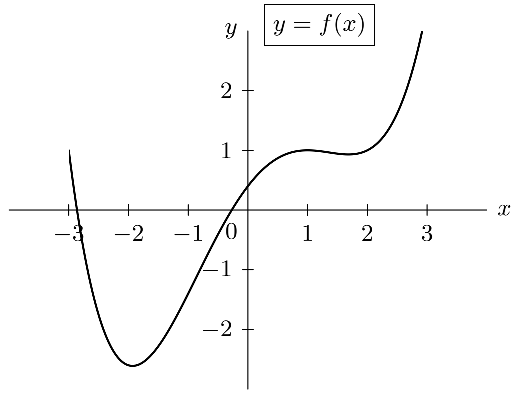
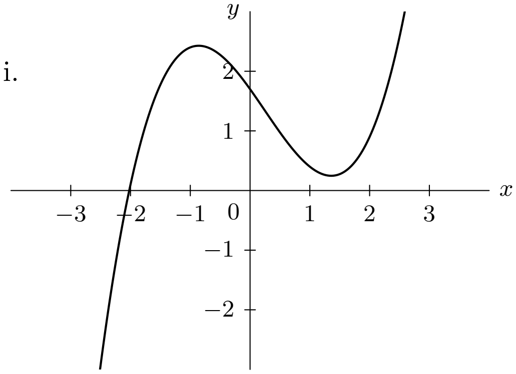
1. [5 points] Below is the graph of a function $f(x)$.

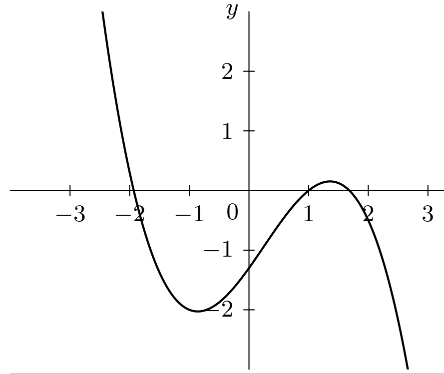


There are six graphs shown below. Circle the <u>one</u> graph that could be the graph of the derivative $f'(x)$.

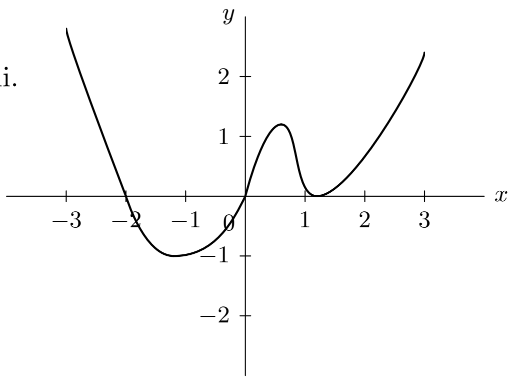
i.



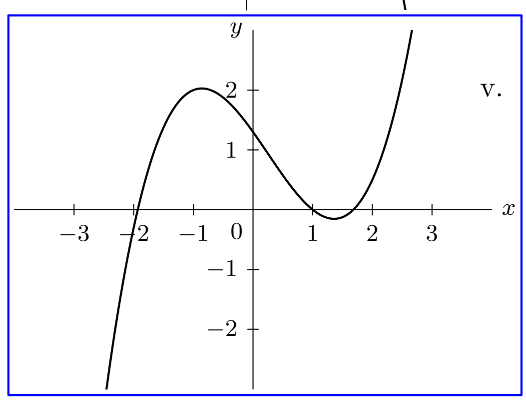
iv.



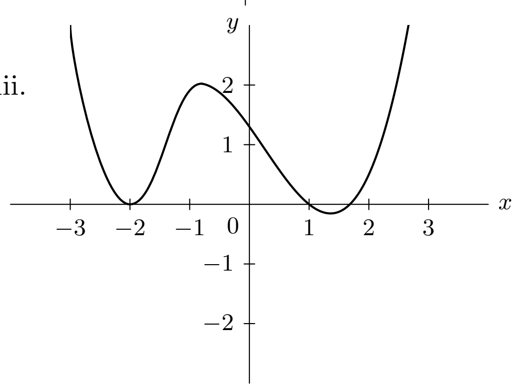
ii.



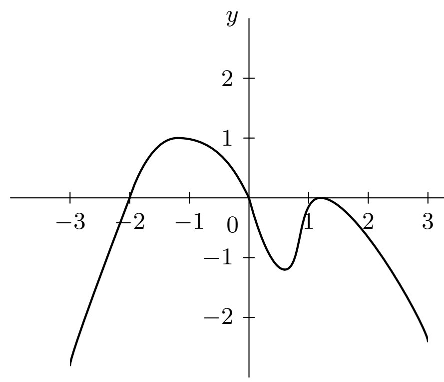
v.



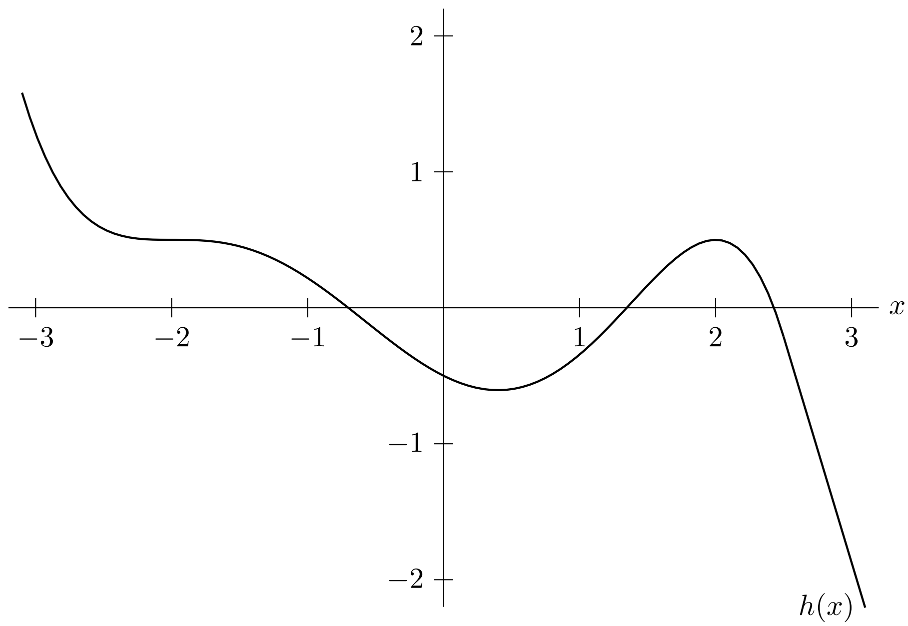
iii.



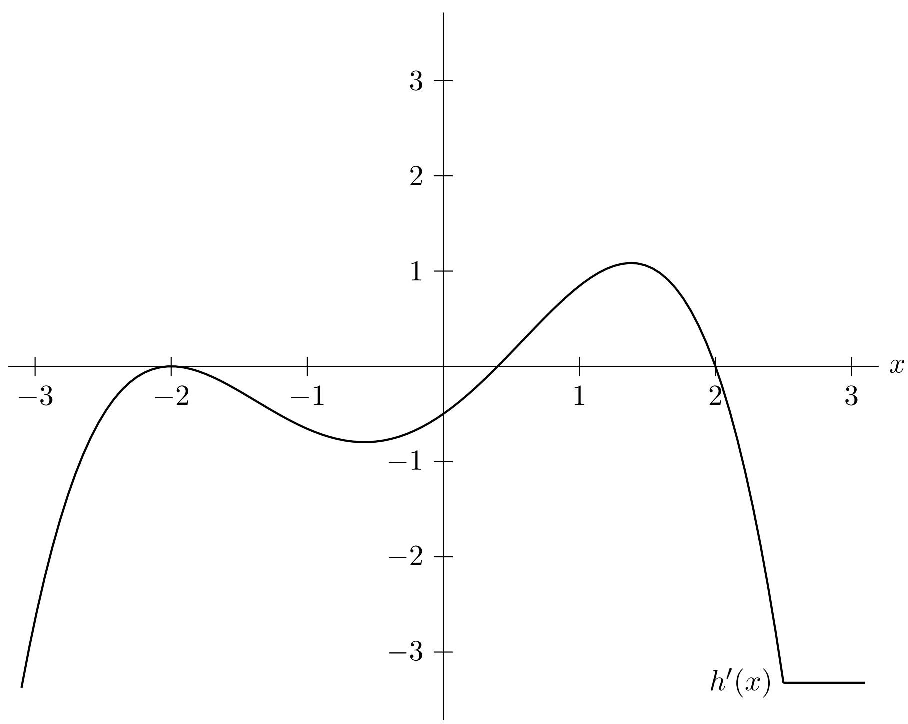
vi.



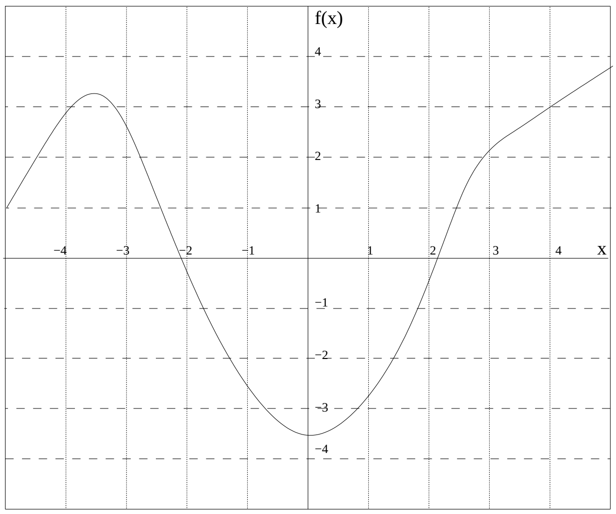
9. [10 points] Given below is the graph of a differentiable function $h(x)$ which is linear for $x > 2.5$. On the second set of axes, sketch a possible graph of $h'(x)$. Be sure your graph is drawn carefully.

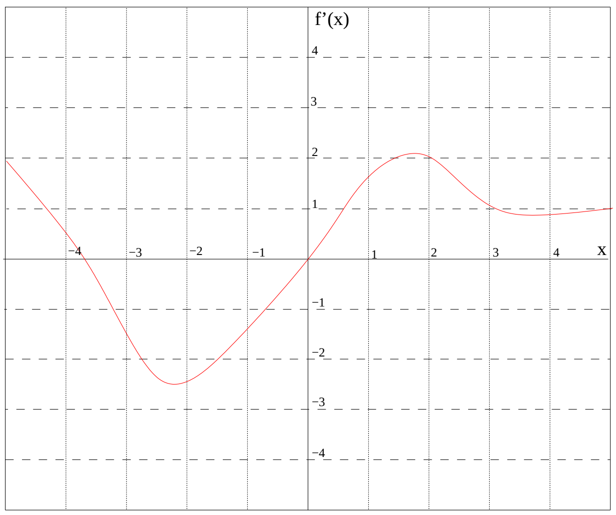


Solution:



(8.) (10 points) The graph of $y = f(x)$ is given in the figure below. On the second set of axes below, sketch the graph of the derivative of f. Use the scale on the graph of f to help estimate values for f'.

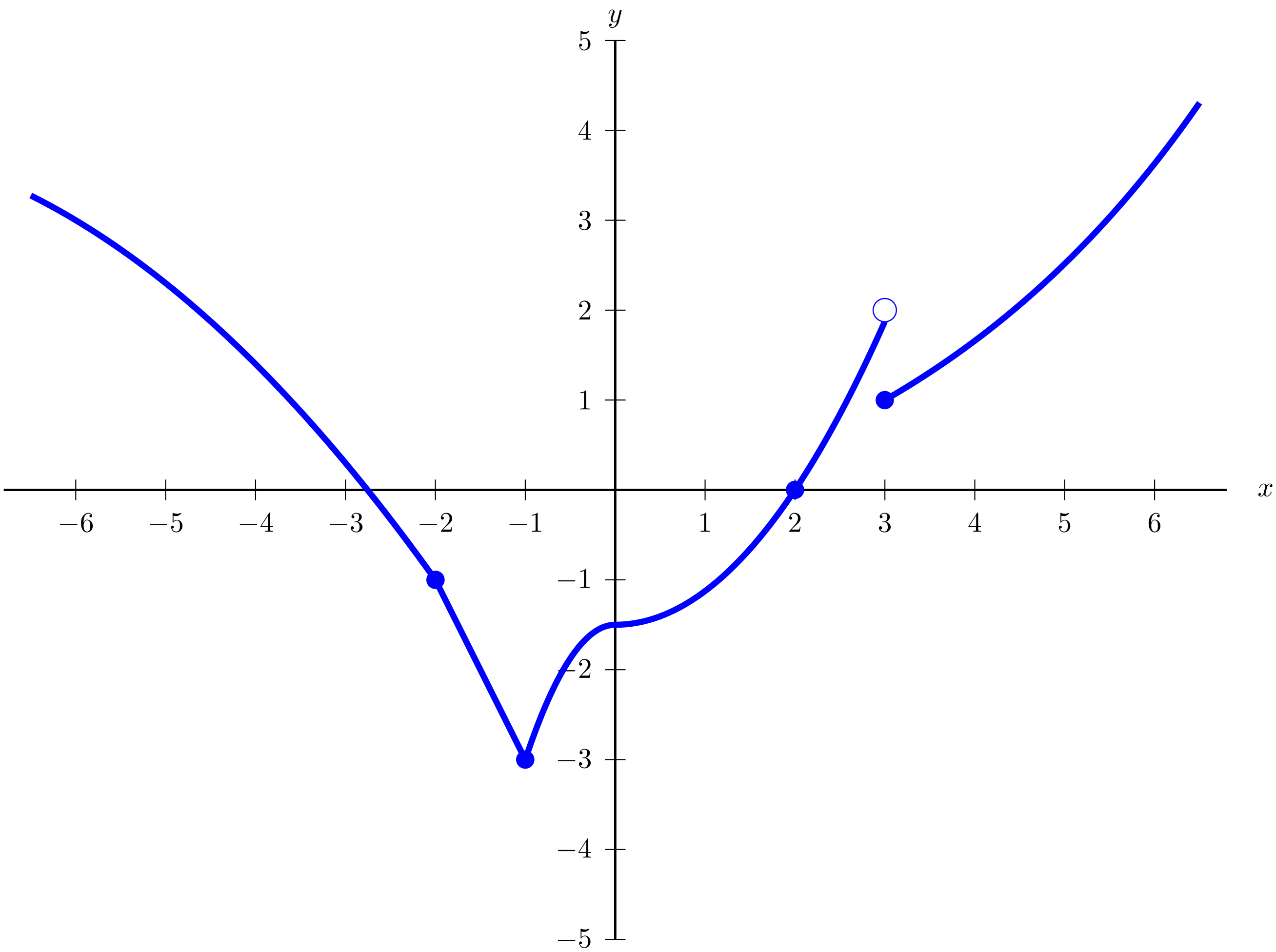




10. [10 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all of the following:

- $h(x)$ is defined for all x in the interval $-6 < x < 6$.
- $h'(x) < 0$ for all $x < -3$.
- $\lim_{x \to -2^+} h(x) = -1$.
- $h'(0) = 0$.
- The average rate of change of $h(x)$ between $x = -1$ and $x = 2$ is 1.
- $h(x)$ is not continuous at $x = 3$.
- $h(x) > 0$ for all $x > 3$.
- $h'(x) > 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.



5. [12 points] A function g defined for all real numbers has the following properties:

(a) g is differentiable for $-1 \leq x < 4$.

(b) $g'(x) \leq 0$ for $-1 \leq x < 4$.

(c) $g''(x) > 0$ for $2 < x < 4$.

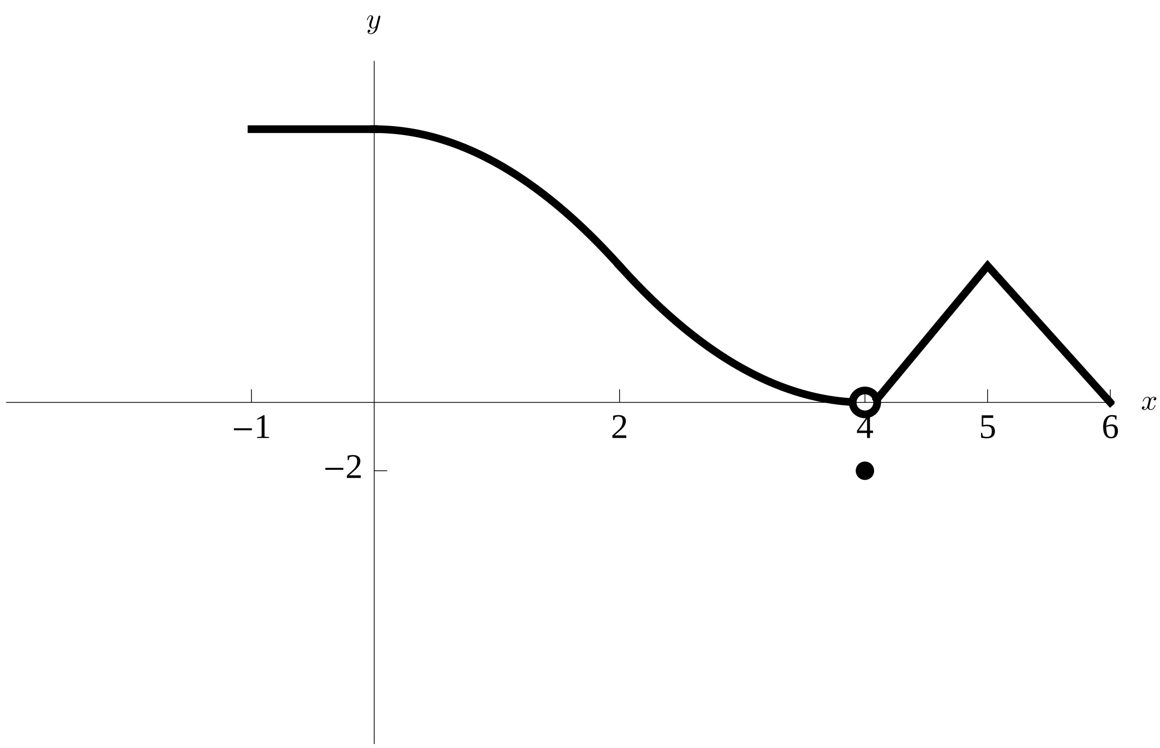
(d) $g(4) = -2$.

(e) $\lim_{x \to 4} g(x) = 0$.

(f) g is continuous at $x = 5$ but not differentiable at $x = 5$.

(g) $g'(0) = 0$.

On the axes below, draw a possible sketch of $y = g(x)$ on the domain $-1 \leq x \leq 6$, including labels.



4. [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

a. [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter P.
Do not attempt to evaluate or simplify the limit.

$$P'(0) = \lim_{h\to 0} \frac{\left((0+h)^2 \sin\left(\frac{1}{0+h}\right) - (0+h)\sin(2)\right) - 0}{h}$$

b. [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
Be sure to include enough clear graphical or numerical evidence to justify your answer.

Solution: We plug in small values of h approaching 0. Since the difference quotient is an even function of h, we need only check positive values of h (as evenness implies that negative h give precisely the same results).

$h = 0.1$:
$$\frac{0.1^2\sin(1/0.1) - 0.1\sin(2) - 0}{0.1} \approx -0.964$$

$h = 0.01$:
$$\frac{0.01^2\sin(1/0.01) - 0.01\sin(2) - 0}{0.01} \approx -0.914$$

$h = 0.001$:
$$\frac{0.001^2\sin(1/0.001) - 0.001\sin(2) - 0}{0.001} \approx -0.908$$

$h = 0.0001$:
$$\frac{0.0001^2\sin(1/0.0001) - 0.0001\sin(2) - 0}{0.0001} \approx -0.909$$

We see at this point that the numbers seem to have stabilized to the nearest hundredth at -0.91.

Answer: $P'(0) \approx$ -0.91

5. [11 points] Oren, a Math 115 student, realizes that the more caffeine he consumes, the faster he completes his online homework assignments. Before starting tonight's assignment, he buys a cup of coffee containing a total of 100 milligrams of caffeine.

Let $T(c)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes c milligrams of caffeine. Suppose that T is continuous and differentiable.

a. [2 points] Circle the ONE sentence below that is best supported by the statement "the more caffeine he consumes, the faster he completes his online homework assignments."

i. $T'(c) \geq 0$ for every value c in the domain of T.

ii. $\boxed{T'(c) \leq 0 \text{ for every value } c \text{ in the domain of } T.}$

iii. $T'(c) = 0$ for every value c in the domain of T.

b. [1 point] Explain, in the context of this problem, why it is reasonable to assume that $T(c)$ is invertible.

Solution: Since the more caffeine Oren consumes the faster he is able to finish his homework, $T(c)$ is a decreasing function. Thus, $T(c)$ is invertible.

c. [2 points] Interpret the equation $T^{-1}(100) = 45$ in the context of this problem. Use a complete sentence and include units.

Solution: In order for Oren to complete his homework assignment in 100 minutes, he must consume 45 milligrams of caffeine.

d. [3 points] Suppose that p and k are constants. In the equation $T'(p) = k$, what are the units on p and k?

Answer: Units on p are **milligrams of caffeine**

Answer: Units on k are **minutes per milligram of caffeine**

e. [3 points] Which of the statements below is best supported by the equation $(T^{-1})'(20) = -10$? *Circle the* ONE *best answer.*

i. If Oren has consumed 20 milligrams of caffeine, then consuming an additional milligram of caffeine will save him about 10 minutes on tonight's assignment.

ii. The amount of caffeine that will result in Oren finishing his homework in 21 minutes is approximately 10 milligrams greater than the amount of caffeine that Oren will need in order to finish his homework in 20 minutes.

iii. The rate at which Oren is consuming caffeine 20 minutes into his homework assignment is decreasing by 10 milligrams per minute.

iv. In order to complete tonight's assignment in 19 rather than 20 minutes, Oren needs to consume about 10 milligrams of additional caffeine.

v. If Oren consumes 20 milligrams of caffeine, then he will finish tonight's assignment approximately 10 minutes faster than if he consumes no caffeine.

3. [10 points] Elphaba the squirrel has been involved in some questionable activity of late and hence is being very cautious. She has made eye contact with a human standing near her multiple times and is getting anxious that the human is observing her. Let $f(x)$ be Elphaba's anxiety (in "anxious units") after making eye contact with the human for a total of x seconds. Elphaba will panic and run when her anxiety reaches 100 anxious units.

From across the room, the human, Erin, is in fact observing Elphaba while pretending to read a newspaper. The total amount of time Elphaba has spent making eye contact with Erin is a function of the number of times that Erin looks up from the newspaper. Let $g(n)$ be the total amount of time, in seconds, that Erin and Elphaba have spent making eye contact if Erin has looked up from her newspaper n times.

a. [2 points] Using a complete sentence, give a practical interpretation of the expression $f^{-1}(3) = 10$. Be sure to include units.

Solution: After Elphaba has made eye contact with Erin for a total of 10 seconds, her anxiety is at 3 anxious units.
Alternative: Elphaba's anxiety is at 3 anxious units when she has made eye contact with Erin for a total of 10 seconds.

b. [3 points] Below is the first part of a sentence that will give a practical interpretation of the equation $f'(25) = 2$.
Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include units in your answer.
If Elphaba has already made eye contact with Erin for a total of 25 seconds and she makes eye contact for an additional 0.3 seconds, then

Solution: If Elphaba has already made eye contact with Erin for a total of 25 seconds and she makes eye contact for an additional 0.3 seconds, then Elphaba's anxiety will increase by approximately 0.6 anxious units.

(Note: $0.6 = 2 \times 0.3$)

c. [2 points] Given that $(f^{-1})'(99) = 7$ and $f(62) = 99$, approximate the total length of time Elphaba has to spend making eye contact with Erin before she will panic and run.

Solution: The expression $(f^{-1})'(99) = 7$ means that once Elphaba's anxiety is at 99 anxiety units, it takes approximately 7 seconds for her anxiety to reach 100 anxious units. The expression $f(62) = 99$ means that it takes 62 seconds for Elphaba's anxiety to reach 99 anxious units. Therefore putting these together, it will take approximately $62 + 7 = 69$ seconds for Elphaba's anxiety to reach 100 anxious units, which is when she will panic and run.

d. [3 points] Which of the following sentences gives a correct interpretation of the quantity $g^{-1}(f^{-1}(50))$? *Circle the* ONE *best answer.*

i. When Erin has looked up from her newspaper 50 times, Elphaba's anxiety is at $g^{-1}(f^{-1}(50))$ anxious units.

ii. When Erin has looked up from her newspaper 50 times, Erin and Elphaba have spent $g^{-1}(f^{-1}(50))$ seconds making eye contact.

iii. If Erin has looked up from her newspaper $g^{-1}(f^{-1}(50))$ times then Elphaba's anxiety is 50 anxious units.

iv. If Erin and Elphaba have made eye contact for a total of 50 seconds then Erin has looked up from her newspaper $g^{-1}(f^{-1}(50))$ times.

v. When Erin and Elphaba have made eye contact for a total of 50 seconds then Elphaba's anxiety is at 50 anxious units.

1. [13 points] For each problem below, circle **ALL** of the statements that **MUST** be true. (The four parts (a)-(d) are independent of each other. No explanations are required.)

a. [3 points] Suppose f is a differentiable function which is concave up on its entire domain, $(-\infty, \infty)$.

- $\lim_{x \to 1} f(x) = f(1)$
- $f(2) \geq f(1)$
- $f'(2) \geq f'(1)$

b. [3 points] Suppose that $h(t)$ gives the height of a ball, measured in feet above ground level, t seconds after it is thrown off a bridge. Assume that the derivative of h is given by the formula $h'(t) = -32t + 64$.

- The ball reaches its maximum height 2 seconds after being thrown.
- The ball reaches a maximum height of 64 feet from the ground.
- The bridge is 64 feet off the ground.

c. [4 points] Suppose that A and B are positive constants and $A < B$.

- $(\ln e^A)(\ln e^B) = A + B$
- $\ln(10^{-A}) < 0$
- $\ln(A^2 + B) = 2 \ln A + \ln B$
- $\log A < \log B$

d. [3 points] Suppose that $f(x) = -Ae^{-Bx}$ for some positive constants A and B.

- $f'(x) > 0$ for all x
- f' is increasing
- f is increasing

5. [12 points] A paperback book (definitely not a valuable calculus textbook, of course) is dropped from the top of Dennison hall (which is 40 m high) towards a very large, upward pointing fan. The average velocity of the book between time $t = 0$ and later times is shown in the table of data below (in which t is in seconds and the velocities are in m/s).

between $t = 0$ seconds and $t =$	1	2	3	4	5
average velocity is	-5	-10	-11.67	-9	-7.2

a. [8 points] Fill in the following table of values for the height $h(t)$ of the book (measured in meters). Show how you obtain your values.

t	0	1	2	3	4	5
$h(t)$	40	**35**	**20**	**5**	**4**	**4**

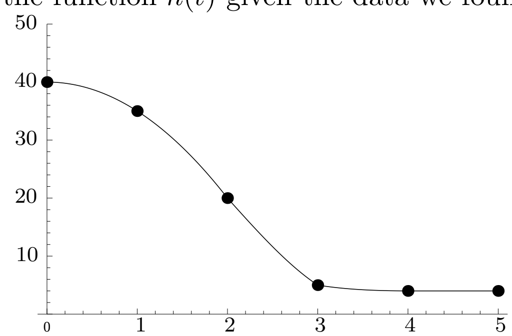
Solution: For each value, we use the definition of average velocity:

$$\text{average velocity on } [0, a] = \frac{h(a) - h(0)}{a}.$$

Thus, the average velocity between $t = 0$ and $t = 1$ gives us $h(1) - 40 = -5$, so $h(1) = 35$. Similarly, between $t = 0$ and $t = 2$ we have $(h(2) - 40)/2 = -10$, so that $h(2) = 20$, etc.

b. [4 points] Based on your work from (a), is $h''(1) > 0$, < 0, or $= 0$? Is $h''(3) > 0$, < 0, or $= 0$? Explain.

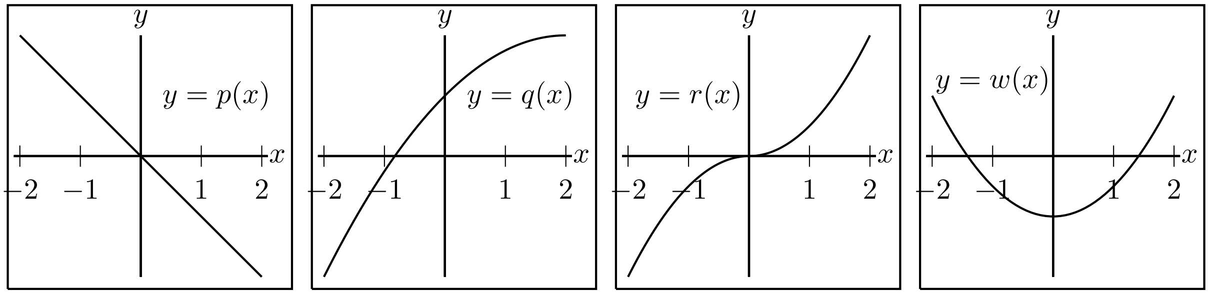
Solution: A sketch of the function $h(t)$ given the data we found in (a) is shown below.



We see that $h(t)$ is concave down at $t = 1$ and concave up at $t = 3$. Thus $h''(1) < 0$ and $h''(3) > 0$.

Alternate solution: The average velocity between $t = 0$ and $t = 1$ is -5 and approximates $h'(0.5)$. The average velocity between $t = 1$ and $t = 2$ is $(20-15)/(2-1) = -15 \approx h'(1.5)$. Thus the velocity appears to be decreasing at $t = 1$, so that $h''(1) < 0$. Similarly we have $h'(2.5) \approx -15$ and $h'(3.5) \approx -1$, so $h''(3) > 0$.

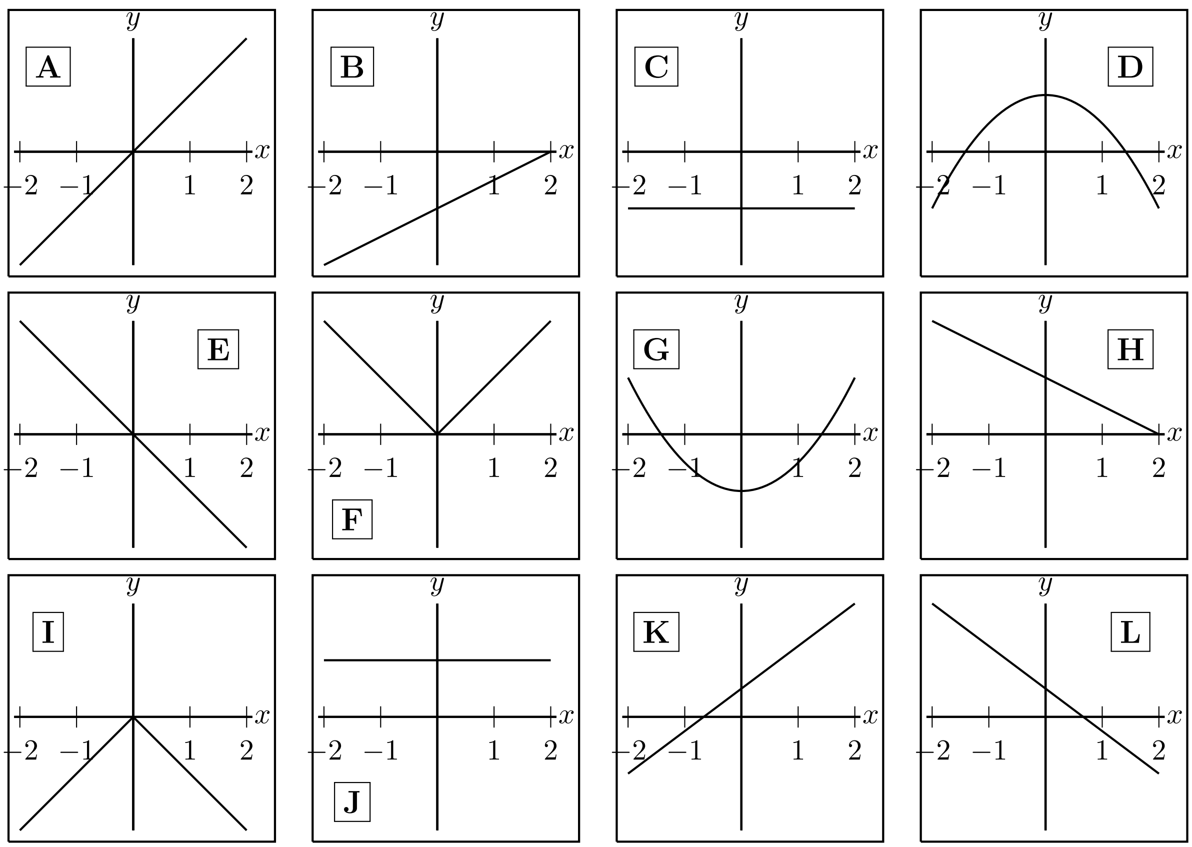
10. [8 points] The graphs of four differentiable functions p, q, r, and w are shown below.



Graph of p': C **Graph of q':** H **Graph of r':** F **Graph of w':** A

For each function above, choose the ONE graph from the choices A-L below that best indicates the behavior of the derivative of that function. Write the capital letter of your choice on the provided answer blank. You may use a letter more than once if necessary. Any unclear answers will be marked as incorrect. (Note that the scale on the y-axis is not indicated on any of the graphs and may vary between the graphs.)

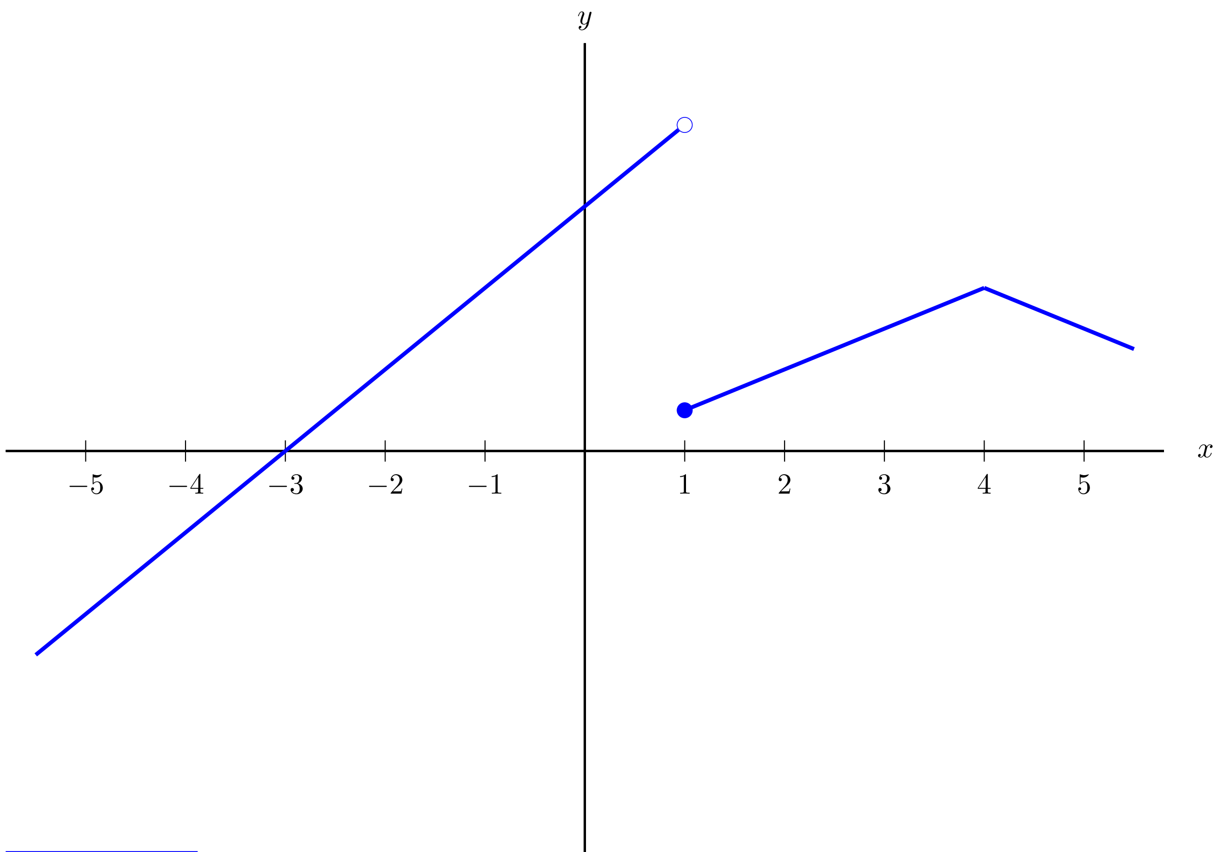
Answer Choices:



8. [8 points] On the axes provided below, sketch the graph of a single function $y = g(x)$ satisfying all of the following:

- $g(x)$ is defined for all x in the interval $-5 < x < 5$.
- $g'(x) > 0$ for all $x < 0$.
- $g(x)$ has a point of discontinuity at $x = 1$.
- The average rate of change of $g(x)$ between $x = -2$ and $x = 2$ is 0.
- $g(x) > 0$ for all $x > 3$.
- $g'(x) < 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.



Solution: Many possibilities exist. Note that in order to satisfy the fourth property, we must have $g(-2) = g(2)$.

9. [3 points] Find all vertical and horizontal asymptotes of the graph of

$$g(x) = \frac{k(x-a)(x-b)}{(x-a)(x-c)^2}$$

where a, b, c, and k are constants with $a < b < c < k$. If there are none, write NONE.

Horizontal asymptote(s): $y = 0$

Vertical asymptote(s): $x = c$

7. [10 points] Sebastian has chartered a helicopter which is rising straight up in the air, but he is scared of heights. Let $A(w)$ be Sebastian's fear (in "scared units") when he is w km above the ground. For $0 < w \leq 2$, a formula for $A(w)$ is given by

$$A(w) = \frac{w^2+2}{w^w+1}.$$

a. [5 points] Use the limit definition of the derivative to write an explicit expression for the instantaneous rate of change of Sebastian's fear, in scared units per km, when he is 1.5 km above the ground. *Your answer should not involve the letter A. Do not attempt to evaluate or simplify the limit.*

Solution: The instantaneous rate of change of Sebastian's fear, in scared units per km, is given by the derivative

$$A'(1.5) = \lim_{h \to 0} \frac{A(1.5+h) - A(1.5)}{h}.$$

Answer: $A'(1.5) = \boxed{\lim_{h \to 0} \dfrac{\dfrac{(1.5+h)^2+2}{(1.5+h)^{(1.5+h)}+1} - \dfrac{1.5^2+2}{1.5^{1.5}+1}}{h}}$

b. [5 points] When he has reached a height of 2 km above the ground Sebastian gets control of his fear and his fear starts decreasing at a constant rate of 0.8 scared units per km. Write a formula for a piecewise-defined continuous function $A(w)$ giving Sebastian's fear, in scared units, for $0 < w < 3$.

Solution: We were given that $A(w) = \frac{w^2+2}{w^w+1}$ for $0 < w \leq 2$. So it remains to find a formula for $A(w)$ that is valid for $2 < w < 3$. Since Sebastian's fear is decreasing at a constant rate for $w > 2$, $A(w)$ is linear for $2 < w < 3$, and the slope of this linear piece is -0.8. In order for $A(w)$ to be continuous, this linear piece must pass through the point $(2, A(2))$ which is $(2, 6/5)$. Using point slope form, this gives the formula $1.2 - 0.8(w-2) = 2.8 - 0.8w$ for the linear piece.

Answer: $A(w) = \begin{cases} \dfrac{w^2+2}{w^w+1} & \text{if } 0 < w \leq 2 \\ 2.8 - 0.8w & \text{if } 2 < w < 3. \end{cases}$

1. [10 points] Laquita decides to visit an amusement park during Fall Break and rides several roller coasters, including the Classic Amazing Looping Coaster and the Ultra Mountain.
Let $R(t)$ be the distance, in feet, that the CAL Coaster has moved along the track t seconds after the ride begins. The ride lasts a total of 60 seconds. Several values of $R(t)$ are shown in the following table.

t	0	10	25	30	40	45	55	60
$R(t)$	0	496	1103	1327	1817	2136	2718	3141

For parts **a.**– **c.**, *remember to show your work and reasoning clearly.*

a. [2 points] Find the average velocity of the CAL Coaster during the last 15 seconds of the ride, i.e. for $45 \le t \le 60$. *Include units.*

Solution: $\dfrac{R(60) - R(45)}{60 - 45} = \dfrac{3141 - 2136}{60 - 45} = \dfrac{1005}{15} = 67.$

Answer: 67 **ft/sec**

b. [2 points] Estimate the instantaneous velocity of the CAL Coaster 30 seconds after the ride begins. *Include units.*

Solution: We estimate using average velocity based on nearby measurements.
Average velocity for $25 \le t \le 30$: $\dfrac{R(30) - R(25)}{30 - 25} = \dfrac{1327 - 1103}{5} = \dfrac{224}{5} = 44.8$
(Note that other answers, such as those incorporating $(40, 1817)$, were accepted if work and appropriate units were shown.)

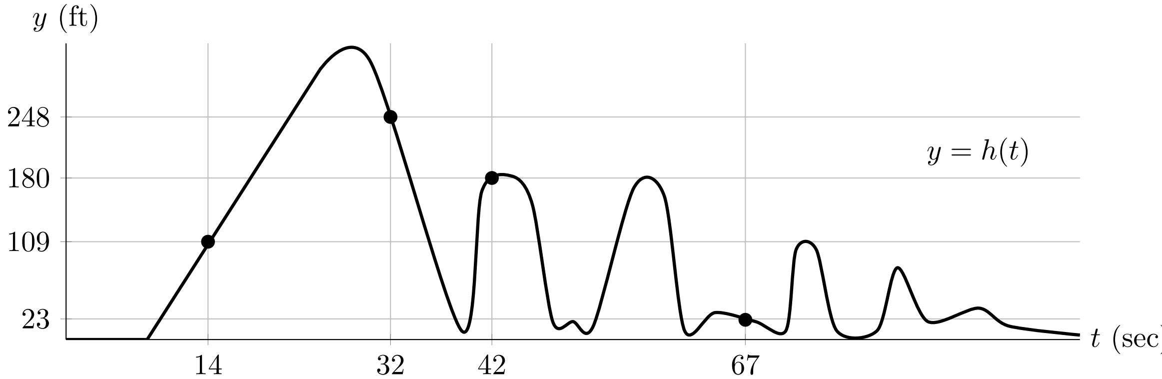
Answer: **About** 44.8 **ft/sec**

c. [2 points] Estimate $R'(55)$.

Solution: We estimate the derivative using the average rate of change of $R(t)$ for $55 \le t \le 60$: $\dfrac{R(60) - R(55)}{60 - 55} = \dfrac{3141 - 2718}{5} = 84.6$
(Again, answers incorporating $(45, 2136)$ were accepted with appropriate work.)

Answer: $R'(55) \approx$ 84.6

d. [4 points] Let $h(t)$ be Laquita's height, in feet, above the ground, t seconds after her ride on the Ultra Mountain begins. A graph of $h(t)$ is shown below.



Let the quantities I–V be defined as follows:

I. The number 0.

II. Laquita's instantaneous vertical velocity, in ft/sec, at $t = 14$.

III. $h'(32)$

IV. Laquita's average vertical velocity, in ft/sec, between $t = 14$ and $t = 42$.

V. Laquita's instantaneous vertical velocity, in ft/sec, at $t = 67$.

Rank the quantities in order from least to greatest by filling in the blanks below with the options I–V. You do not need to show your work.

III < **V** < **I (0)** < **IV** < **II**