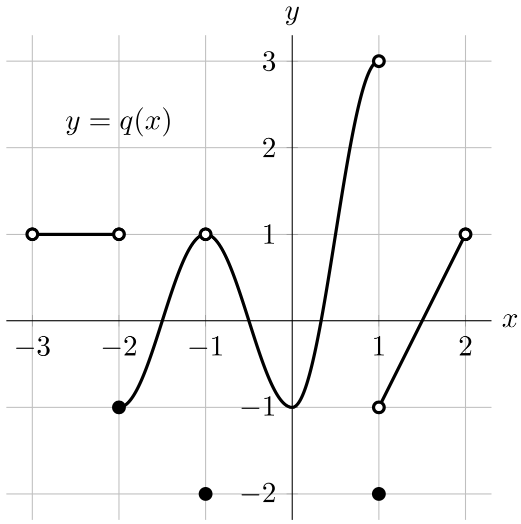
3. [10 points] The entire graph of a function q is shown below. Note that $q(x)$ is linear on the interval $1 < x < 2$.



Throughout this problem, you do not need to explain your reasoning.

For each of parts **a.**– **c.** below, circle all of the listed values satisfying the given statement. If there are no such values, circle NONE.

a. [2 points] For which of the following values of a does $\lim_{t\to a} q(t)$ exist?

$a = -2$ $\boxed{a = -1}$ $\boxed{a = 0}$ $a = 1$ NONE

b. [2 points] For which of the following values of b is $q(x)$ continuous at $x = b$?

$b = -2$ $b = -1$ $\boxed{b = 0}$ $b = 1$ NONE

c. [2 points] For which of the following values of c is $\lim_{x\to c^+} q(x) = q(c)$?

$\boxed{c = -2}$ $c = -1$ $\boxed{c = 0}$ $c = 1$ NONE

For each of parts **d.** and **e.** below, if the limit does not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE.

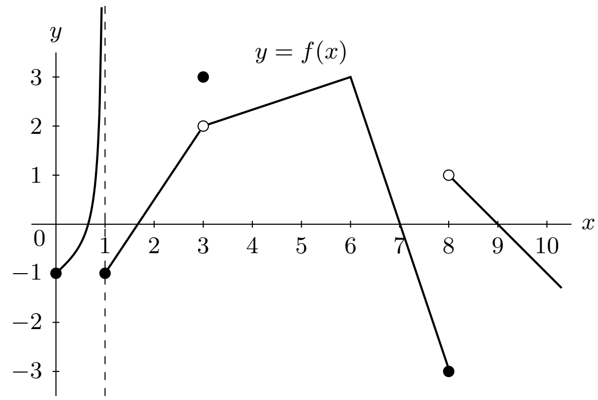
d. [2 points] Evaluate the following expression: $\lim_{k\to 0} \dfrac{q(1.21 + k) - q(1.21)}{k}$.

Answer: 2

e. [2 points] Evaluate the following expression: $\lim_{s\to -1} q(q(s))$.

Answer: 3

8. [12 points] A portion of the graph of a function f is shown below.



a. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x\to c} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ 1, 8

b. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x\to c^+} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ **NONE**

c. [2 points] Give all values c in the interval $0 < c < 10$ for which $f(x)$ is not continuous at c. If there are none, write NONE.

Answer: $c =$ 1, 3, 8

d. [6 points] With f as shown in the graph above, define a function g by the formula

$$g(x) = \begin{cases} \dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} & \text{if } x \le 0 \\ f(x) & \text{if } 0 < x < 10 \end{cases}$$

where A and B are nonzero constants.
Find values of A and B so that both of the following conditions hold.

- $g(x)$ is continuous at $x = 0$.
- $\lim_{x\to-\infty} g(x) = \dfrac{1}{2}$.

If no such values exist, write NONE in the answer blanks.
Be sure to show your work or explain your reasoning.

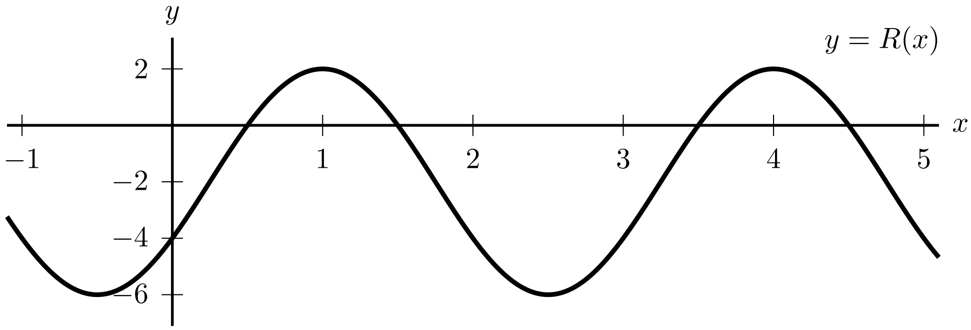
Solution: To satisfy the first condition, we first compute $g(0)$ by plugging in $x = 0$ to the rational function $\dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5}$ to find $\lim_{x\to 0-} g(x) = g(0) = \dfrac{B}{12}$. In order for $g(x)$ to be continuous at $x = 0$, we must also have $\lim_{x\to 0^+} g(x) = \dfrac{B}{12}$.
Now $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} f(x) = -1$ (from the graph), so $\frac{B}{12} = -1$, and $B = -12$.
To satisfy the second condition, we compute that

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} = \frac{A}{4}.$$

In order for this limit to equal $\frac{1}{2}$, we must have $A/4 = 1/2$, so $A = 2$.

Answer: $A =$ 2 and $B =$ -12

8. [6 points] Given below is the graph of a sinusoidal function $R(x)$.



Find a possible formula for $R(x)$.

Solution: The graph shown above is of a sinusoidal function with amplitude 4, period 3, and midline $y = -2$. We first consider the graph of

$$y = 4\cos\left(\frac{2\pi}{3}x\right) - 2\,.$$

This graph has the proper amplitude, period, and midline. We shift this graph over to the right 1 unit to obtain the graph of $y = R(x)$. Thus, one possible formula for $R(x)$ is given by

$$R(x) = 4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2.$$

Answer: $R(x) =$ $4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2$

9. [4 points] The table below gives several values of a function $w(x)$.

x	4.5	4.9	4.99	5	5.01	5.1	5.5
$w(x)$	-0.879	-0.154	-0.015	0	0.060	0.630	3.750

Use the information in the table above to estimate the following limit.

$$\lim_{h\to 0^-} \frac{w(5+h)}{h}$$

Clearly show any computations that you use to make this estimate.

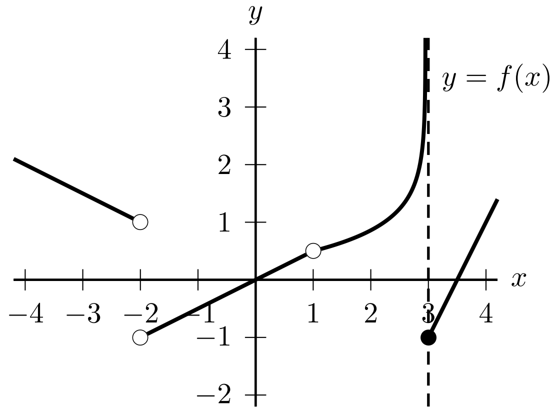
Solution: The left limit can be approximated by $w(5+h)/h$ for small negative values of h. The table of values provided for w allows us to compute this when $h = -0.1$ and when $h = -0.01$. The results are shown in the table below.

h	$\frac{w(5+h)}{h}$
-0.1	$\frac{w(4.9)}{-0.1} = 1.54$
-0.01	$\frac{w(4.99)}{-.01} = 1.5$

Using these values, we estimate that the desired left-hand limit is approximately 1.5.

Answer: $\lim_{h\to 0^-} \frac{w(5+h)}{h} \approx$ 1.5

6. [11 points] Below is the graph of a portion of a function $f(x)$.



a. [2 points] Give all values of a in the interval $-4 < a < 4$ that are not in the domain of $f(x)$. If there are none, write NONE.

Answer: -2, 1

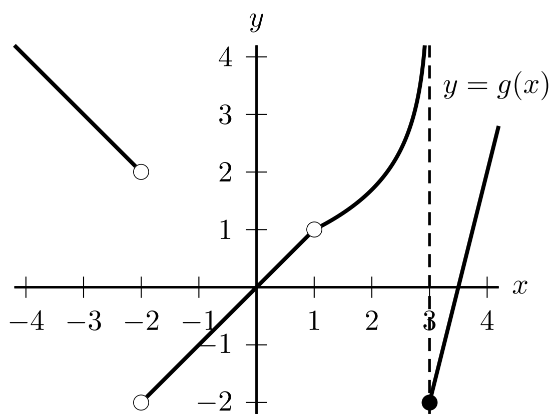
b. [2 points] Give all values of a in the interval $-4 < a < 4$ where $f(x)$ is not continuous at $x = a$. If there are none, write NONE.

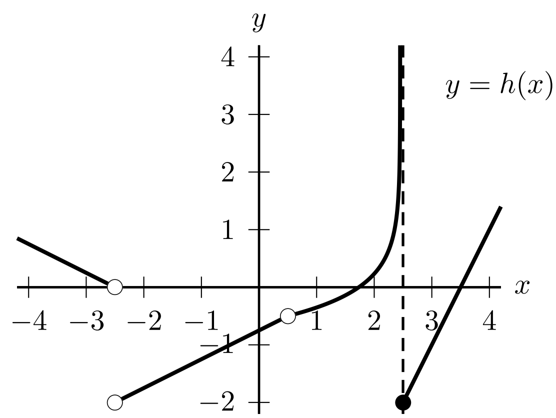
Answer: -2, 1, 3

c. [2 points] Give all values of a in the interval $-4 < a < 4$ where $\lim_{x \to a} f(x)$ does not exist. If there are none, write NONE.

Answer: -2, 3

d. [5 points] The graphs below show portions of two other functions $g(x)$ and $h(x)$ which are transformations of $f(x)$. Express $g(x)$ and $h(x)$ as transformations of $f(x)$.





Answer: $g(x) =$ $2f(x)$ and $h(x) =$ $f(x + 0.5) - 1$

4. [10 points] For each of the following, give a *formula* for a single function satisfying all of the listed properties. If there is no function satisfying all the properties, circle NO SUCH FUNCTION EXISTS.

Note: If "NO SUCH FUNCTION EXISTS" is circled, then any formula you have written will not be graded.

a. [3 points] A *polynomial* $p(t)$ with the following three properties:

- The degree of $p(t)$ is three.
- $p(t) \to -\infty$ as $t \to \infty$.
- $p(0) = -4$.

Solution: Note that the second property implies that the leading coefficient of the polynomial is negative, and the third property implies that, when written in standard form, the constant term of $p(t)$ is -4. So one example is $p(t) = -t^3 - 4$.

Answer: $p(t) =$ $-t^3 - 4$ OR Circle: NO SUCH FUNCTION EXISTS

b. [3 points] An *exponential function* $q(v)$ with the following three properties:

- $q(1) = 3$.
- $\lim\limits_{v \to 0} q(v) = 12$.
- $\lim\limits_{v \to \infty} q(v) = 0$.

Solution: Since exponential functions are continuous, the second property implies that $q(0) = 12$. So $q(v)$ is exponential with initial value 12 and decay factor equal to $\frac{q(1)}{q(0)} = \frac{3}{12} = \frac{1}{4}$. Therefore q must be the function given by $q(v) = 12\left(\frac{1}{4}\right)^v$.

Answer: $q(v) =$ $12\left(\dfrac{1}{4}\right)^v$ OR Circle: NO SUCH FUNCTION EXISTS

c. [4 points] A *rational* function $r(x)$ with the following three properties:

- The line $x = 2$ is a vertical asymptote of the graph of $y = r(x)$.
- The line $y = -3$ is a horizontal asymptote of the graph of $y = r(x)$.
- $r(5) = 0$.

Solution: These properties imply that $r(x)$ can be written as a quotient of polynomials $\frac{p(x)}{q(x)}$ such that $(x-2)$ is a factor of $q(x)$, the ratio of the leading term of $p(x)$ to that of $q(x)$ is -3, and $(x-5)$ is a factor of $p(x)$. There are many possibilities, but one example is $r(x) = \frac{-3(x-5)}{x-2}$.

Answer: $r(x) =$ $\dfrac{-3(x-5)}{x-2}$ OR Circle: NO SUCH FUNCTION EXISTS

2. [12 points] A scientist is growing a very large quantity of mold. Initially, the mass of mold grows exponentially, but after many hours, the mass stabilizes at 24 kilograms.
Suppose that t hours after the scientist begins, the mass of mold, in kilograms, can be modeled by the function M defined by the equation

$$M(t) = \begin{cases} 0.41e^{0.72t} & \text{if } 0 \le t \le 5 \\ \dfrac{2t^3}{at^b + c} & \text{if } t > 5. \end{cases}$$

a. [4 points] Find the value of k between 0 and 5 so that $M(k) = 1$. Then interpret the equation $M(k) = 1$ in the context of this problem. Use a complete sentence and include units.

Solution: Because we want to find k between 0 and 5, we use the first piece of the formula for M and solve for k in the equation $0.41e^{0.72k} = 1$.

$$\begin{aligned} 0.41e^{0.72k} &= 1 \\ e^{0.72k} &= 1/0.41 \approx 2.439 \\ 0.72k &= \ln(1/0.41) \approx 0.892 \\ k &= \ln(1/0.41)/0.72 \approx 1.238 \end{aligned}$$

Answer: $k = \frac{\ln(1/0.41)}{0.72} \approx 1.238$

Interpretation:

Solution: 1.238 hours after the scientist begins, the mold has a mass of 1 kg.

b. [8 points] Assuming that M is a continuous function of t, determine $\lim_{t\to\infty} M(t)$, and find the values of a, b, and c.

Solution:

$$\frac{2}{a} = 24 \text{ so } a = 1/12 \approx 0.083$$

$$\begin{aligned} 0.41e^{0.72 \cdot 5} &= \frac{2 \cdot 5^3}{5^3/12 + c} \\ 0.41e^{3.6} &= \frac{250}{125/12 + c} \\ \frac{125}{12} + c &= \frac{250}{0.41e^{3.6}} \\ c &= \frac{250}{0.41e^{3.6}} - \frac{125}{12} \approx 6.244 \end{aligned}$$

Answers: $\lim_{t\to\infty} M(t) = 24$ $\quad a = 1/12 \approx 0.083$

$b = 3$ $\quad c = \frac{250}{0.41e^{3.6}} - \frac{125}{12} \approx 6.244$

10. [9 points] Suppose data is collected at a U-M basketball game held at Crisler Center. Let $E(t)$ be the total amount of electricity, in megawatt-hours (MWh), that has been used by Crisler Center during the first t minutes of the basketball game, which starts at exactly 7:00 pm. Assume that E is invertible and that both E and E^{-1} are differentiable.

a. [3 points] Suppose b and c are positive constants. Use a complete sentence to give a practical interpretation of the equation

$$E(30 + b) = E(30) + c$$

in the context of this problem. Your sentence should involve the constants b and c but not "E". Be sure to include units.

Solution: In the b minutes after 7:30 pm, Crisler Center uses c MWh of electricity.

b. [3 points] Fill in the two answer blanks below to write a single mathematical equality involving the derivative of either E or E^{-1} which supports the following claim:

"During the basketball game, Crisler Center uses about 1.8 MWh of electricity during the first 3 seconds after 7:45 pm."

Answer: $E'(45)$ = 36

c. [3 points] Which of the sentences below best expresses the meaning of the equation

$$E^{-1}(20) = 1.5E^{-1}(12)$$

in the context of this problem? (Circle the one best choice.)

A. Crisler Center uses 50% more electricity during the first 20 minutes after the game starts than during the first 12 minutes after the game starts.

B. It takes half as long for Crisler Center to use the first 12 MWh of electricity during the game than for it to use the next 8 MWh.

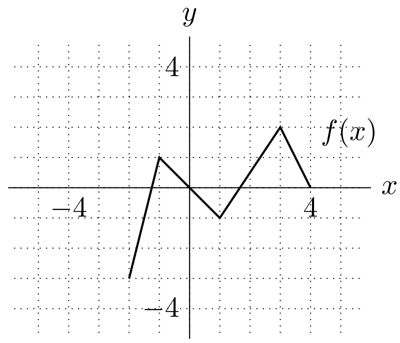
C. Crisler Center uses 50% as much electricity during the first 20 minutes after the game starts than during the first 12 minutes after the game starts.

D. It takes 50% longer for Crisler Center to have used a total of 20 MWh of electricity during the game than for it to use the first 12 MWh.

E. Crisler Center uses twice as much electricity during the first 20 minutes after the game starts than during the next 12 minutes.

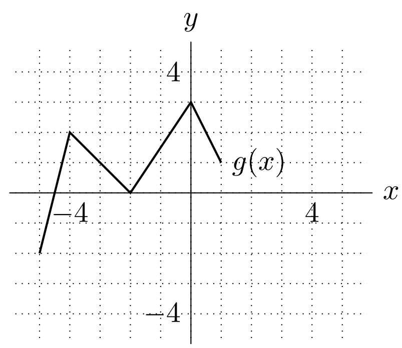
F. It takes 50% less time for Crisler Center to have used a total of 12 MWh of electricity during the game than for it to use the first 20 MWh.

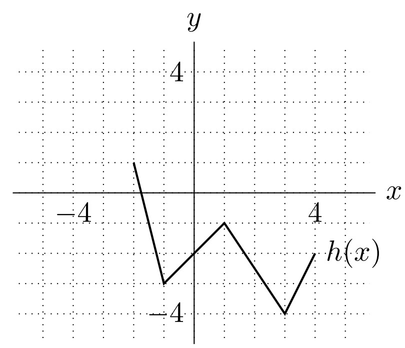
7. [15 points] The graph of a function $f(x)$ is shown below. The domain of $f(x)$ is $-2 \leq x \leq 4$.



You do not need to show work on this page.

a. [6 points] Each of the functions $g(x)$ and $h(x)$ shown below is a transformation of the function $f(x)$. Write a formula for each function in terms of $f(x)$.





$g(x) =$ $f(x+3)+1$ $h(x) =$ $-f(x)-2$

b. [4 points] Determine the domain and range of the function $j(x) = -2f(x-6)+3$.

Domain: **4** $\leq x \leq$ **10** Range: **-1** $\leq y \leq$ **9**

c. [5 points] On the axes below, draw a graph of the derivative of $f(x)$.