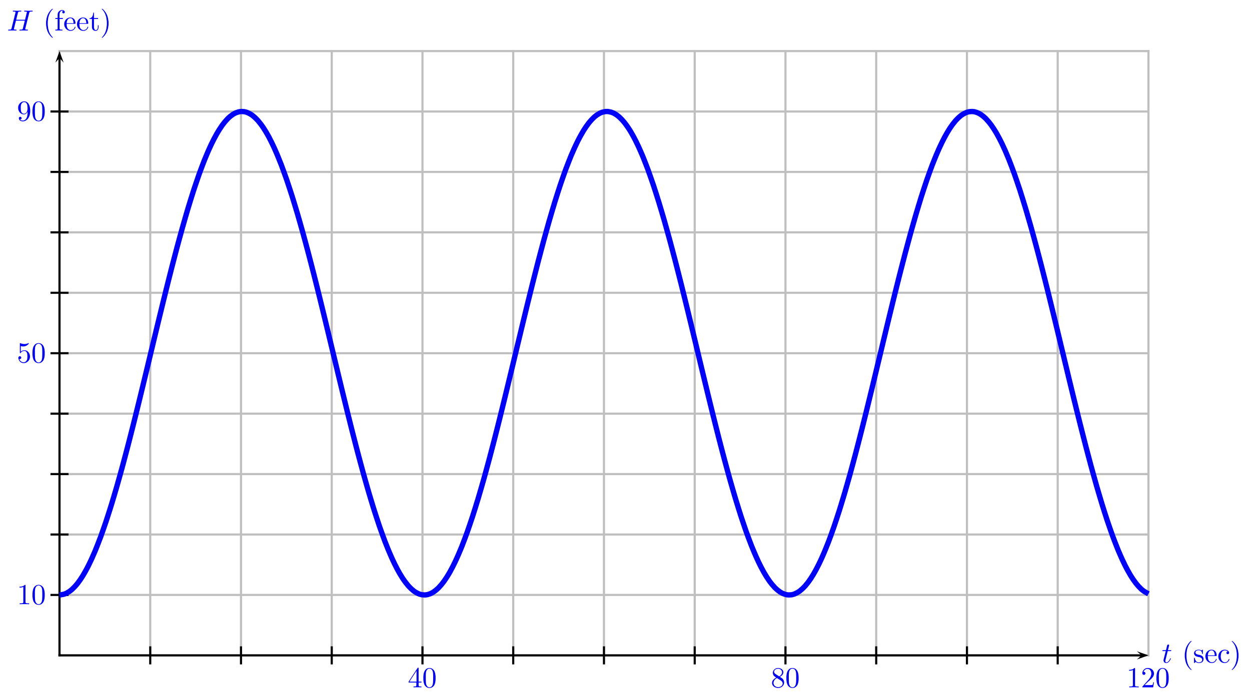
6. [12 points] At the county fair, there is a ferris wheel with radius 40 feet. Riders board at the lowest point of the ferris wheel, from a platform 10 feet off the ground. Once the ride begins, the ferris wheel completes 3 revolutions in 120 seconds. Suppose that you are the last rider to board (so you begin at the lowest point), and the function $H(t)$ measures your height off the ground (in feet), t seconds after the ride starts.

a. [4 points] On the grid below, sketch a graph $H(t)$ for one complete ride (3 revolutions). Be sure to carefully label the axes.



b. [4 points] Find the period and amplitude of $H(t)$.

Solution: 40 seconds

period = ________________________

Solution: 40 feet

amplitude = ________________________

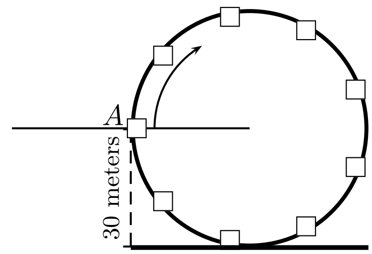
c. [4 points] Find a formula for $H(t)$.

Solution:

$$H(t) = 50 - 40\cos\left(\frac{\pi}{20}t\right)$$

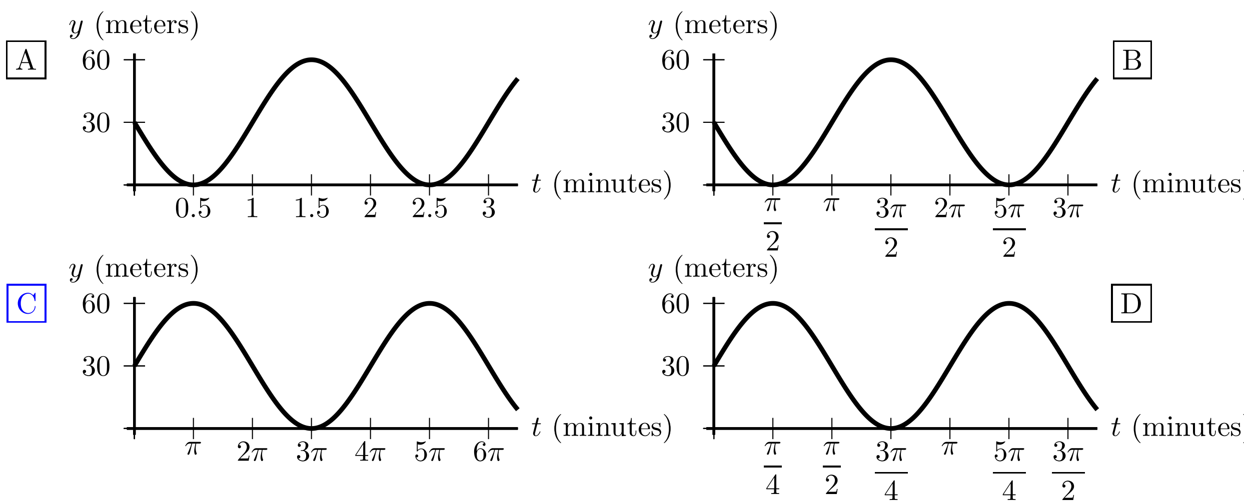
6. [10 points] Erin is in pursuit of squirrel and suspected criminal, Elphaba. Suddenly there is a cliff ahead. Elphaba, who it turns out is a flying squirrel, jumps straight off and glides safely down to the ground. Searching for an alternative, Erin finds a ferris wheel that will take her to the ground beneath the cliff.

The ferris wheel has radius 30 meters and is rotating (clockwise in the diagram shown) at a constant rate of one half radian per minute. Let $H(t)$ be Erin's height above the ground beneath the cliff (in meters) t minutes after she gets on the ferris wheel. A diagram of the situation is shown to the right. Note that Erin gets on the ferris wheel at position A, and $H(0) = 30$.



a. [2 points] Which of the following graphs is a graph of $y = H(t)$?
Write the letter (A-D) of the ONE *best choice.* **Answer:** C

Solution: Erin is 30 meters above the ground at time $t = 0$, and her height above the ground increases at first. Since the ferris wheel is rotating at a constant rate of $1/2$ radian per minute, it will take 4π minutes for it to rotate 2π radians. So the period of $H(t)$ is 4π. The graph shown as $\boxed{\text{C}}$ below is the best choice.



b. [4 points] Write a formula for the sinusoidal function $H(t)$.

Solution: $H(t)$ is a sinusoidal function with amplitude 30, midline $y = 30$, and period 4π. The value is on the midline and increasing when $t = 0$. There are many possible formulas. One is $H(t) = 30\sin(\frac{t}{2}) + 30$.

Answer: $H(t) =$ $30\sin\left(\frac{t}{2}\right) + 30$

c. [4 points] Erin figures that if she jumps off when she is no more than b meters above the ground, where b is a constant between 0 and 30, then she will be fine. Erin would like to jump off before she has to go around the ferris wheel again. What is the latest time she can jump off without going around a full revolution? Remember to show your work clearly. Your answer may involve the constant b.

Solution: We are looking for a particular solution to the equation $30\sin(t/2) + 30 = b$. Using inverse trig to solve this, we find that one solution is $t = 2\sin^{-1}(\frac{b-30}{30})$. Now we need to think about what point on the graph this actually gives us. In this case, since $-1 < \frac{b-30}{30} < 0$ this solution is the first solution to the left of the vertical axis in the graph above. We need to add the period 4π to get the time we are looking for. So to avoid going around a full revolution, the latest time Erin can jump off is $2\sin^{-1}\left(\frac{b-30}{30}\right) + 4\pi$ minutes after she got on the ferris wheel.

Answer: $t = 2\sin^{-1}\left(\frac{b-30}{30}\right) + 4\pi$

10. [4 points] Find <u>all</u> real numbers B and positive integers k such that the rational function

$$H(x) = \frac{9 + x^k}{16 - Bx^3}$$

satisfies the following two conditions:

- $H(x)$ has a vertical asymptote at $x = 2$
- $\lim_{x\to\infty} H(x)$ exists.

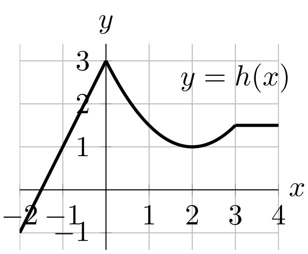
If no such values exist, write NONE.

Justification: In order for $\lim_{x\to\infty} H(x)$ to exist, the degree of the polynomial in the numerator has to be smaller or equal to 3 (the degree of the polynomial in the denominator). In order for $x = 2$ to be a vertical asymptote, you need the denominator to be zero. Hence $16 - B(2^3) = 16 - 8B = 0$ which requires $B = 2$. In this case $H(x) = \dfrac{9 + x^k}{16 - 2x^3}$ with $k = 1, 2$ or 3. Since $9 + 2^k \neq 0$, then $H(x)$ has a vertical asymptote at $x = 2$ when $B = 2$.

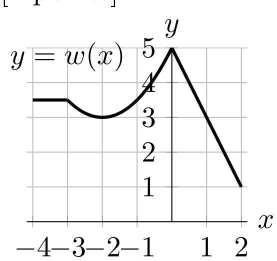
Answer: $B =$ **2** **Answer:** $k =$ **1, 2, or 3**

11. [4 points] A part of the graph of a function $h(x)$ is given below.

In each of the following parts, the corresponding portion of the graph of a function obtained from h by one or more transformations is shown, together with a list of possible formulas for that function. In each case choose the one correct formula for the function shown. *Note that the graphs are not all drawn at the same scale.*



a. [2 points]



A. $h(-x) - 2$

B. $-h(x) - 2$

C. $-h(x) + 2$

D. $\boxed{h(-x) + 2}$

E. $-h(x + 2)$

F. $-h(x - 2)$

G. $-h(-x + 2)$

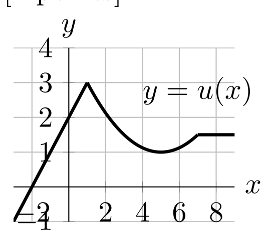
H. $-h(-x - 2)$

I. $h(-x + 2)$

J. $h(-x - 2)$

K. NONE OF THESE

b. [2 points]



A. $h(0.5x + 1)$

B. $h(0.5x - 1)$

C. $\boxed{h(0.5(x - 1))}$

D. $h(0.5(x + 1))$

E. $h(0.5(x - 0.5))$

F. $h(2x - 1)$

G. $h(2x + 1)$

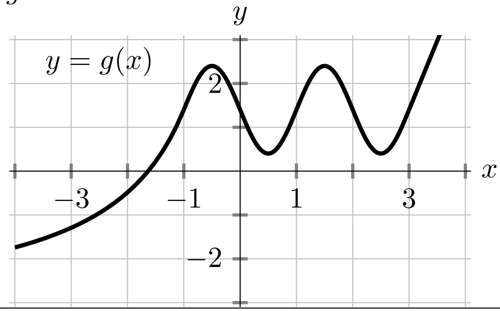
H. $h(2(x - 1))$

I. $h(2(x + 1))$

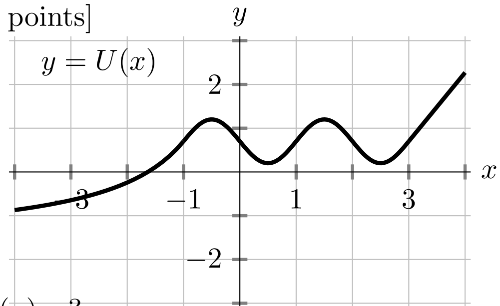
J. $h(0.5(x + 0.5))$

K. NONE OF THESE

11. [11 points] A portion of the graph of a function g is shown below. In each of parts **a.–d.** on this page, the corresponding portion of the graph of a function obtained from g by one or more transformations is shown, together with a list of possible formulas for that function. In each case, circle the one correct formula for the function shown.



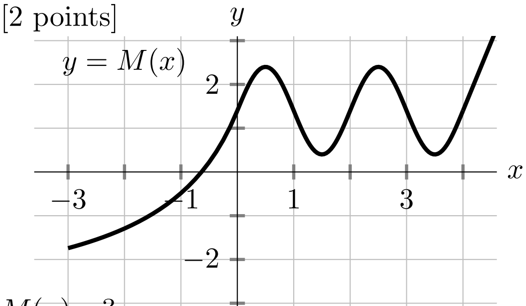
a. [2 points]



$U(x) = ?$
Circle the one correct choice below.

$g(x) - 1$	$g(0.5x)$	$\boxed{0.5g(x)}$
$g(x) + 1$	$g(2x)$	$2g(x)$
$g(x) - 1.5$	$g(x + 1)$	$g(x - 1)$

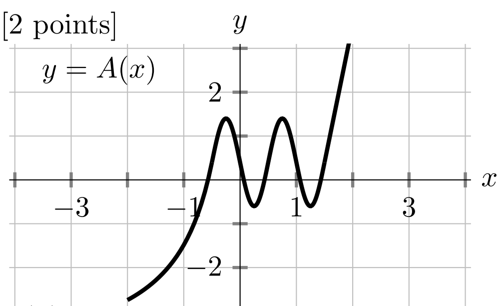
b. [2 points]



$M(x) = ?$
Circle the one correct choice below.

$g(x) - 1$	$g(0.5x)$	$0.5g(x)$
$g(x) + 1$	$g(2x)$	$2g(x)$
$g(x) - 1.5$	$g(x + 1)$	$\boxed{g(x - 1)}$

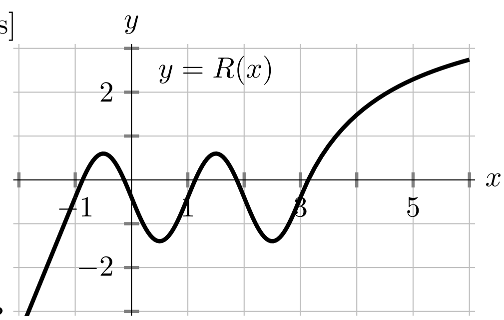
c. [2 points]



$A(x) = ?$
Circle the one correct choice below.

$g(2x) + 1$	$g(0.5x) + 1$	$g(x - 2) - 1$
$\boxed{g(2x) - 1}$	$g(0.5x) - 1$	$2g(x - 1)$
$2g(x + 1)$	$0.5g(x + 1)$	$0.5g(x - 1)$

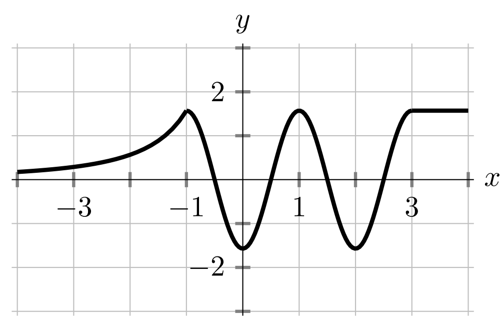
d. [2 points]



$R(x) = ?$
Circle the one correct choice below.

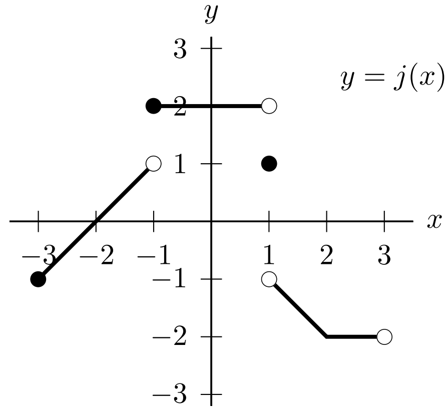
$g(-x - 1) + 2$	$-g(x - 1) - 2$	$-g(x + 2) - 1$
$g(-x + 1) - 2$	$-g(-x - 2) - 1$	$-g(x - 2) + 1$
$g(-x - 2) + 1$	$\boxed{-g(-x + 2) + 1}$	$-g(-x + 1) + 2$

e. [3 points] A portion of the graph of the derivative of one of the five functions above is shown on the right. Which derivative is shown? Circle the one correct choice below.

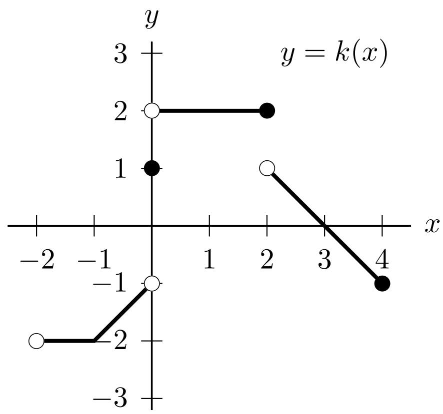


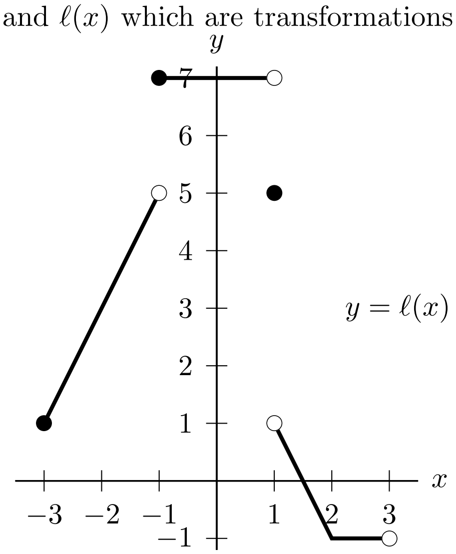
$g'(x)$ $\boxed{U'(x)}$ $M'(x)$ $A'(x)$ $R'(x)$

11. [6 points] Below is the graph of a function $j(x)$.



The graphs below show two other functions $k(x)$ and $\ell(x)$ which are transformations of $j(x)$. Write a formula for each in terms of j and x.





Answer: $k(x) =$ $j(-(x-1))$ and $\ell(x) =$ $2j(x) + 3$

12. [3 points] Find a formula for one polynomial $p(x)$ that satisfies both of the following properties.

- The degree of $p(x)$ is at least 5.
- The domain of the function $\ln(p(x))$ is the interval $(-\infty, \infty)$.

Note that this problem may have more than one correct answer. You only need to find one correct answer.

Solution: Since the domain of $\ln(x)$ is the interval $(0, \infty)$, a number x is in the domain of $\ln(p(x))$ if and only if $p(x) > 0$. So we need to find a polynomial of degree at least 5 that is always positive. Note that any such polynomial has to have even degree (since the end behavior of an odd degree polynomial differs on the two ends). One possible answer is $p(x) = x^6 + 1$.

Answer: $p(x) =$ $x^6 + 1$

7. [9 points] Enjoying breakfast outdoors in a coastal Mediterranean town, Tommy notices a ship that is anchored offshore. The ship is stationed above a reef which lies below the surface of the water, and a series of waves causes its height to oscillate sinusoidally with a period of 6 seconds. When Tommy begins observing, the hull of the ship is at its highest point, 20 feet above the reef. After 1.5 seconds, the hull is 11 feet above the reef.

a. [6 points] Write a function $h(t)$ that gives the height of the ship's hull above the reef t seconds after Tommy begins observing.

Solution:

$$h(t) = 9\cos\left(\frac{\pi}{3}t\right) + 11.$$

The function starts out at its maximum, so we will use cosine with no horizontal shift making our formula $h(t) = A\cos(Bt) + C$. The period is given to be 6. This means $B = 2\pi/6 = \pi/3$. When $t = 1.5$, we have

$$11 = h(1.5) = A\cos\left(\frac{\pi}{3}\cdot 1.5\right) + C = C$$

and so when $t = 0$ we have

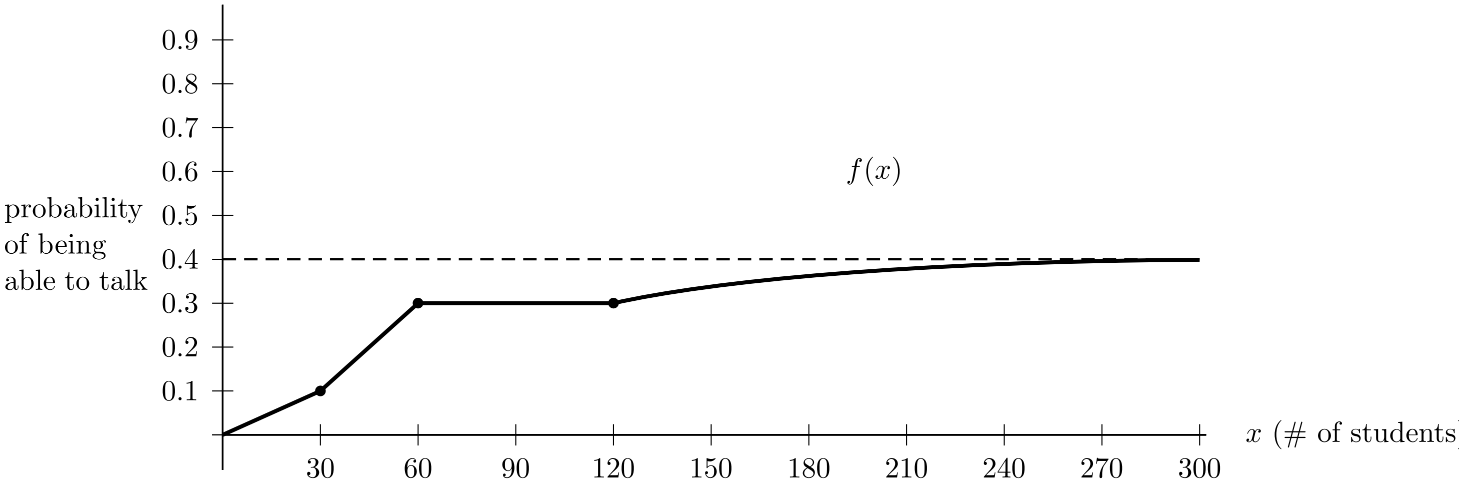
$$20 = h(0) = A + C = A + 11.$$

Solving this, we have $A = 9$.

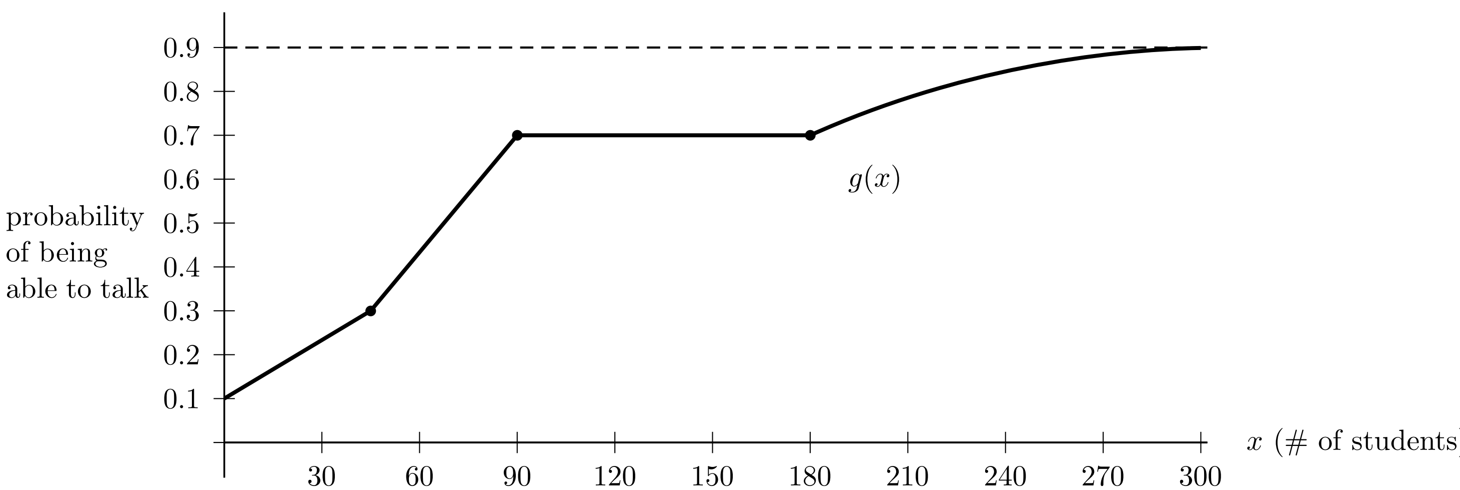
b. [3 points] According to your function, will the hull of the ship hit the reef? Explain.

Solution: No. At it's lowest point at $t = 3$, $h(3) = 2$, so the ship remains two feet above the reef.

9. [12 points] Lauren has just been approached by a talking kangaroo named Skipper. Lauren is alarmed by Skipper's ability to talk so she asks him if all kangaroos can talk. Skipper tells her that a kangaroo's probability of being able to talk depends on how much attention they receive from University of Sydney students. He also explains that this relationship has changed recently. Let $f(x)$ be the probability that a kangaroo born prior to 5 weeks ago is able to talk if x students have paid attention to that kangaroo in their life. Below is a graph of the function $f(x)$.



a. [4 points] Let $g(x)$ be the probability that a kangaroo born within the last 5 weeks is able to talk if x students have paid attention to that kangaroo in their life. A graph of $g(x)$ is given below.



It turns out that for $x \geq 0$, $g(x)$ can be expressed as a transformation of $f(x)$. Write a formula for $g(x)$ as a transformation of $f(x)$.

Solution: To obtain the graph of $g(x)$ from that of $f(x)$, we first vertically stretch by a factor of 2, then shift the resulting graph up by 0.1 units, and finally stretch it horizontally by a factor of 3/2.

Answer: $g(x) = 2f\left(\frac{2}{3}x\right) + 0.1$

Problem continues on the next page.

This is a continuation of the problem from the previous page.

b. [3 points] Let $h(y)$ be the number of students that pay attention to a kangaroo over the duration of the kangaroo's life if the kangaroo has y docile units. (Note that we measure docileness in terms of docile units.) Give a practical interpretation of $g(h(5))$.

Solution: $g(h(5))$ is the probability that a kangaroo born in the last 5 weeks with 5 docile units will be able to talk.

c. [3 points] The number of students who pay attention to a kangaroo with y docile units is proportional to y^2. Find a formula for $h(y)$ if a total of 160 students pay attention to a kangaroo with 4 docile units during its life.

Solution: The first sentence tells us that $h(y) = ky^2$ for some constant k. Using the fact that $h(4) = 160$, we find $160 = k(4^2)$, so $160 = 16k$ and $k = 10$.

Answer: $h(y) =$ $10y^2$

d. [2 points] Calculate $f(h(3))$.

Solution: Using our formula from part c, we have $h(3) = 10(3^2) = 90$ so $f(h(3)) = f(90) = 0.3$.

Answer: $f(h(3)) =$ 0.3

4. [10 points] Consider the function f defined by $f(x) = \dfrac{(x+1.8)(x+2.1)}{(2x+1.8)(3x-6.9)(x+2.1)}$.

You do not have to show your work/reasoning on this problem. However, any work that you do show may be considered for partial credit.

a. [3 points] What is the domain of f?

Answer: **all real numbers except -0.9, 1.3, and -2.1**

b. [2 points] Find the equations of all vertical asymptotes of the graph of $y = f(x)$. If there are none, write NONE.

Answer: $x = -0.9$ **and** $x = 1.3$

c. [2 points] Let $g(x) = e^{-0.4x}$.

Find the equations of all horizontal asymptotes of the graph of $y = \dfrac{g(x)}{f(x)}$.

If there are none, write NONE.

Solution: $g(x)$ is a positive exponential decay function and dominates any rational function as $x \to \infty$. In particular, $\lim\limits_{x\to\infty} \dfrac{g(x)}{f(x)} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{g(x)}{f(x)} = \infty$ (DNE), so the only horizontal asymptote of the graph of $y = \dfrac{g(x)}{f(x)}$ is $y = 0$.

Answer: $y = 0$

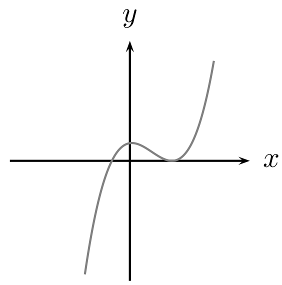
d. [3 points] Find a formula for a rational function $h(x)$ such that $\lim\limits_{x\to\infty} \dfrac{f(x)}{h(x)} = 8$.

Solution: There are many possible answers. Some examples include:

- $h(x) = \dfrac{1}{8 \cdot 6 \cdot x} = \dfrac{1}{48x}$, and
- $h(x) = \dfrac{1}{8} f(x) = \dfrac{(x+1.8)(x+2.1)}{8(2x+1.8)(3x-6.9)(x+2.1)}$.

Answer: $h(x) = \dfrac{1}{48x}$

7. [8 points] For each of the graphs below, select the formula beneath the graph which best fits the behavior of the graph. In each case, assume that A, B, C, D, E, F, and G are positive constants. (Circle your choice. No work or explanation is necessary.)

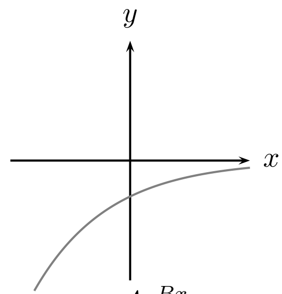


$y = A(x-B)(x+C)$

$\boxed{y = A(x-B)^2(x+C)}$

$y = -A(x+B)^2(x-C)$

$y = A(x+B)^2(x-C)$

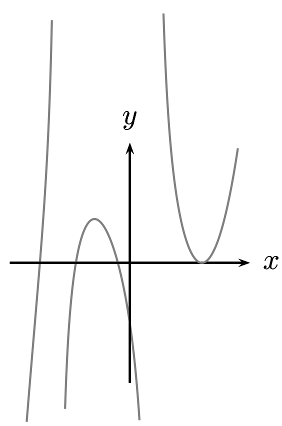


$y = Ae^{Bx}$

$y = Ae^{-Bx}$

$y = -Ae^{Bx}$

$\boxed{y = -Ae^{-Bx}}$

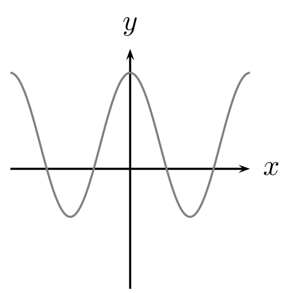


$$y = \frac{A(x-B)(x-C)(x-D)(x+E)^2}{(x+F)(x-G)}$$

$$y = \frac{A(x+B)(x+C)(x+D)(x-E)}{(x+F)(x-G)}$$

$$\boxed{y = \frac{A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)}}$$

$$y = \frac{-A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)^2}$$



$y = -B\cos(Cx) - A$

$\boxed{y = A + B\cos(Cx)}$

$y = -A + B\sin(Cx + D)$

$y = A - B\sin(Cx)$