3. [12 points] Consider the family of linear functions

$$L(x) = ax - 3$$

and the family of functions

$$M(x) = a\sqrt{x}$$

where a is a nonzero constant number. Note that the number a is the same for both equations. Find a value of a for which $L(x)$ is tangent to the graph of $M(x)$. Also find the x and y coordinates of the point of tangency. Write your answers in the blanks provided.

Solution: Since the graphs of M and L intersect are tangent at the relevant point, we have

$$M'(x) = L'(x). \qquad (\dagger)$$

Computing the derivatives, we get

$$\frac{1}{2}ax^{-\frac{1}{2}} = a$$

Canceling the a from this equation and doing some algebra, we see $x = \frac{1}{4}$. Since the two graphs must also intersect at this point, we must have

$$M\left(\frac{1}{4}\right) = L\left(\frac{1}{4}\right),$$

and using this, we can solve for a; we get $a = -12$. Finally, we can recover $y = -6$ by plugging these values into either the equation for $M(x)$ or the equation for $L(x)$.

$a =$ **-12**

$x =$ **.25**

$y =$ **-6**

9. [13 points] Consider the function

$$f(x) = ax \ln x - bx$$

with domain $x > 0$, where a and b are positive constants. Note that this function has exactly one critical point.

a. [3 points] Find $f'(x)$.

Solution:

$$f'(x) = a\left((1)\ln x + x\frac{1}{x}\right) - b = a\ln x + a - b$$

b. [4 points] For which values of a and b does $f(x)$ have a critical point at $(e, -2)$?

Solution: We want $f'(e) = 0$ and $f(e) = -2$. We plug $x = e$ into $f'(x)$ to get

$$f'(e) = a\ln e + a - b = 2a - b = 0$$

which tells us that $2a = b$. Plugging $x = e$ into $f(x)$ gives

$$f(e) = ae\ln e - be = (a - b)e = -2.$$

Using $2a = b$, we get

$$(a - (2a))e = -2$$

so that $a = 2/e$. Since $b = 2a$, we get $b = 4/e$.

c. [3 points] Using your values of a and b from part **(b)**, is the critical point from **(b)** a local maximum, local minimum, or neither? Justify your answer.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. Then $f''(e) = 4/e^2 > 0$ so the graph of $f(x)$ is concave up at $x = e$. This means that $f(x)$ has a local minimum at $x = e$.

d. [3 points] Using your values of a and b from part **(b)**, find the x-coordinates of any inflection points of $f(x)$ or show that $f(x)$ has no inflection points.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. This is continuous and positive for all values of x in the domain $x > 0$ of $f(x)$. Since the second derivative never changes sign, $f(x)$ has no inflection points.

7. [13 points] Consider the family of functions

$$y = ax^b \ln x$$

where a and b are nonzero constants.

a. [4 points] Calculate $\frac{dy}{dx}$ in terms of the constants a and b.

Solution: $\frac{dy}{dx} = abx^{b-1}\ln x + ax^b\left(\frac{1}{x}\right) = abx^{b-1}\ln x + ax^{b-1} = ax^{b-1}(b\ln x + 1)$.

b. [9 points] Find specific values of a and b so that the resulting function has a local maximum at the point $(e, 1)$. You must show that $(e, 1)$ is a local maximum to receive full credit.

Solution: The point $(e, 1)$ must be on the curve, so $1 = ae^b \ln e = ae^b$. So $a = e^{-b}$.
We also know $(e, 1)$ is a critical point, so

$$0 = ae^{b-1}(b\ln e + 1) = ae^{b-1}(b - 1).$$

$a \neq 0$ and $e^{b-1} \neq 0$ so $b + 1 = 0$, which means $b = -1$ and $a = e^{-(-1)} = e$.
To check that $(e, 1)$ is a local maximum, we use the second derivative test:
First, we plug in the values of a and b to our derivative and get $\frac{dy}{dx} = ex^{-2}(1 - \ln x)$. So we have

$$\frac{d^2y}{dx^2} = e(-2)x^{-3}(1 - \ln x) + ex^{-2}\left(\frac{-1}{x}\right).$$

Now we can plug in $x = e$ and get $-2e \cdot e^{-3}(1 - \ln e) - e \cdot e^{-3} = -e^{-2} < 0$. Thus, $(e, 1)$ is a local maximum.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \le b \le 100 \\ 4b + 200 & \text{for } 100 < b \le 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: \$6

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: \$4

For $0 \le b \le 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: \$20

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Solution: Oren's marginal revenue is $R'(b) = 6$ for $0 < b < 100$ and $R'(b) = 4$ for $100 < b < 160$. His marginal cost is $C'(b) = 3 + 12/\sqrt{b}$.
Thus, $R'(b) = C'(b)$ for $b = 16$ and $b = 144$.

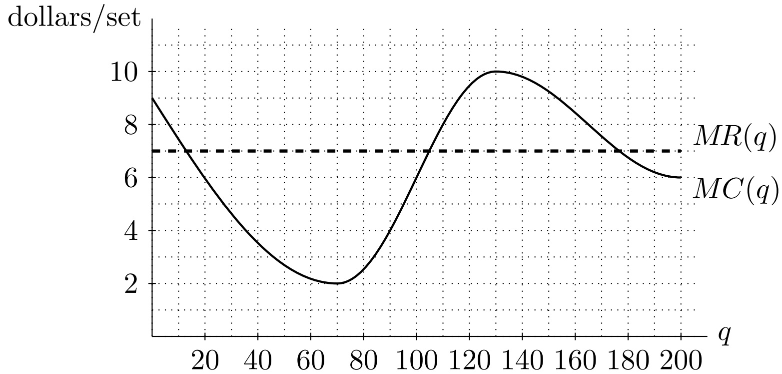
Answer: **at 16 bunches and 144 bunches**

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Since Oren's profit function, $\pi(b) = R(b) - C(b)$, is continuous on $0 \le b \le 160$, it has a global maximum (by the Extreme Value Theorem) and the global maximum occurs at a critical point or an endpoint.
The critical points of $\pi(b)$ occur when $\pi'(b) = 0$ (at $b = 16$ and 144 (when MR=MC)), and when $\pi'(b)$ is undefined (at $b = 100$).
We check the value of $\pi(b)$ at the critical points and end points:
$\pi(0) = -20$, $\pi(16) - -68$, $\pi(100) = 40$, $\pi(144) = 36$, and $\pi(160) \approx 36.42$, and conclude that the maximum occurs at $b = 100$, with a resulting maximum profit of \$40.

Answer: bunches of kale: 100 and max profit: \$40

2. [12 points] Link has started a business selling winter clothes for cats. Among his most successful products are his new kitten mittens. He is currently selling his mittens for \$7 per set. Below is a graph of Link's marginal cost $MC(q)$ and marginal revenue $MR(q)$, in dollars per set of mittens, if he makes q sets of mittens this winter. Due to a shortage of yarn, Link can make a maximum of 200 sets of mittens this winter. In order to start making mittens, Link must spend \$40 on knitting supplies (in other words, it costs \$40 to make 0 sets of mittens).



You do not need to show any work for this problem.

a. [3 points] Approximately how many sets of mittens should Link make this winter in order to maximize his profit?

Answer: Link should make about **104** sets of mittens.

b. [2 points] If the price per set is raised to \$9, approximately how many sets of mittens should Link make in order to maximize his profit?

Answer: Link should make about **200** sets of mittens.

c. [3 points] Write an expression involving integrals which equals Link's total profit if Link makes 150 sets of mittens. Your expression may involve the functions $MR(q)$ and $MC(q)$.

Solution:

$$\int_0^{150} (MR(q) - MC(q))\, dq - 40$$

d. [4 points] Link makes a deal with a store that would like to buy his cat hats. If the store buys up to 50 hats, then each one will cost \$10. If the store buys more than 50 hats, then Link will reduce the price of the entire order by \$0.05 per hat for every additional hat over 50. (For example, if the store buys 52 hats, they will pay \$9.90 per hat.) Write a formula for a function $L(q)$ which gives Link's revenue if he sells q hats to the store.

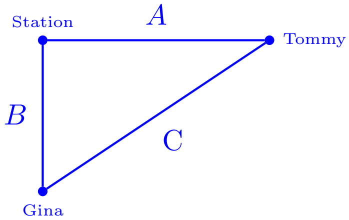
$$L(q) = \begin{cases} 10q & \text{if } 0 \leq q \leq 50 \\ (10 - 0.05(q-50))q & \text{if } q > 50 \end{cases}$$

5. [10 points] Tommy and Gina were friends in high school but then went to college in different parts of the country. They thought they were going to see each other in Springfield over the December break, but their schedules didn't match up. In fact, it turns out that Tommy is leaving on the same day that Gina is arriving.

Shortly before Gina's train arrives in Springfield, she sends a text to Tommy to see where he is, and Tommy sends a text response to say that, sadly, his train has already left. At the moment Tommy sends his text, he is 20 miles due east of the center of the train station and moving east at 30 mph while Gina is 10 miles due south of the train station and moving north at 50 mph.

a. [2 points] What is the distance between Gina and Tommy at the time Tommy sends his text? *Remember to include units.*

Solution:



Let A, B, and C be the distances (in miles) as labeled in the drawing on the left. Then by the Pythagorean Theorem, $C^2 = A^2 + B^2$. When Tommy sends his text, $A = 20$ and $B = 10$ so we conclude that Gina and Tommy are $\sqrt{20^2 + 10^2} = \sqrt{500} = 10\sqrt{5} \approx 22.4$ miles apart.

Answer: $\sqrt{500} = 10\sqrt{5} \approx 22.4$ **miles**

b. [6 points] When Tommy sends his text, are he and Gina moving closer together or farther apart? How quickly? *You must show your work clearly to earn any credit. Remember to include units.*

Solution: With the notation from part (a), we have that

$$2C\frac{dC}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}.$$

When Tommy sends his text, we know that $\frac{dA}{dt} = 30$ and $\frac{dB}{dt} = -50$. Thus,

$$\frac{dC}{dt} = \frac{A\frac{dA}{dt} + B\frac{dB}{dt}}{C} = \frac{(20)(+30) + (10)(-50)}{\sqrt{500}} = 2\sqrt{5} \approx 4.47.$$

Since at this time the sign of $\frac{dC}{dt}$ is positive, Gina and Tommy are getting farther apart.

Answer: Tommy and Gina are getting (circle one) CLOSER TOGETHER **[FARTHER APART]**

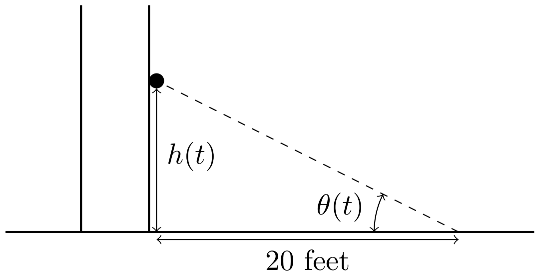
at a rate of $2\sqrt{5} \approx 4.47$ **mph**

c. [2 points] Let $J(t)$ be the distance between Gina and Tommy t hours after Tommy sends his text. Use the local linearization of $J(t)$ at $t = 0$ to estimate the distance between Gina and Tommy 0.1 hours after Tommy sends his text. *Remember to show your work carefully.*

Solution: This local linearization is $L(t) = J(0) + J'(0)(t) = 10\sqrt{5} + 2t\sqrt{5}$. So the distance between Tommy and Gina 0.1 seconds after Tommy sends his text will be approximately $L(0.1) = 10\sqrt{5} + (0.1)(2\sqrt{5}) = 10.2\sqrt{5} \approx 22.8$ miles.

Answer: $10.2(\sqrt{5}) \approx 22.8$ **miles**

6. [12 points] Walking through Nichols Arboretum, you see a squirrel running down the trunk of a tree. The trunk of the tree is perfectly straight and makes a right angle with the ground. You stop 20 feet away from the tree and lie down on the ground to watch the squirrel. Suppose $h(t)$ is the distance in feet between the squirrel and the ground, and $\theta(t)$ is the angle in radians between the ground and your line of sight to the squirrel, with t being the amount of time in seconds since you stopped to watch the squirrel.



a. [3 points] Write an equation relating $h(t)$ and $\theta(t)$. (Hint: Use the tangent function.)

Solution:

$$\tan\theta(t) = \frac{h(t)}{20}$$

b. [5 points] If $\theta(t)$ is decreasing at 1/5 of a radian per second when $\theta(t) = \pi/3$, how fast is the squirrel moving at that time?

Solution: Differentiate with respect to t:

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

Plug in $\theta'(t) = -1/5$ and $\theta(t) = \pi/3$:

$$\frac{1}{\cos^2(\pi/3)}(-1/5) = \frac{h'(t)}{20}$$

Solve to get $h'(t) = -16$ so the squirrel is moving at -16 feet per second.

c. [4 points] For the last second before the squirrel reaches the ground, it is moving at a constant speed of 20 feet per second. Suppose $\theta'(t) = -3/4$ at some point during this last second. How high is the squirrel at this time?

Solution: Start with

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

and plug in $h'(t) = -20$ and $\theta'(t) = -3/4$:

$$\frac{1}{\cos^2\theta(t)}(-3/4) = \frac{-20}{20}$$

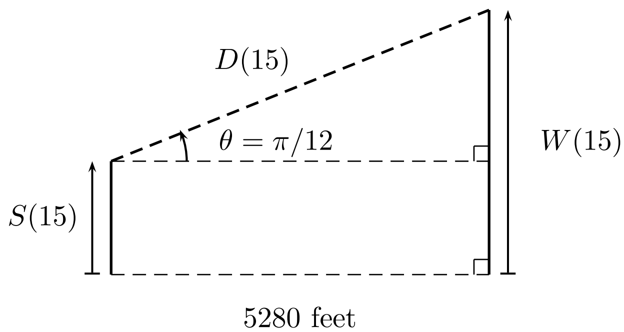
This gives us that $\cos^2\theta(t) = 3/4$. Then $\cos\theta(t) = \sqrt{3}/2$ (positive because $0 < \theta < \pi/2$), and $\theta(t) = \arccos(\sqrt{3}/2) = \pi/6$. Finally, we use the equation from part (a) to get

$$h(t) = 20\tan(\pi/6) = 20/\sqrt{3} \approx 11.547$$

so the squirrel is at a height of about 11.547 feet.

9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



Solution: Since $\cos(\pi/12) = \frac{5280}{D(15)}$, we find that $D(15) = \frac{5280}{\cos(\pi/12)} \approx 5466.258\text{ft}$.

b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

Solution: From the previous part, we know that the distance between the ship and the whale is 5280 divided by the cosine of the angle you've turned your head. Since $\theta(t)$ gives how far you've turned your head, we can find the distance at any time t using the function $D(t) = \frac{5280}{\cos(\theta(t))}$.

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

Solution: Since $D(t) = \frac{5280}{\cos(\theta(t))}$, we take the derivative with respect to t on both sides to get

$$\frac{dD}{dt} = \frac{5280\sin(\theta(t))}{\cos^2(\theta(t))}\theta'(t).$$

Since $\theta(15) = \pi/12$ and we are given $\theta'(15) = .01$, we get

$$\left.\frac{dD}{dt}\right|_{t=15} = \frac{5280\sin(\pi/12)}{\cos^2(\pi/12)}(.01) \approx 14.6468\text{ft/sec}.$$

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

Solution: The right triangle in the figure above has hypotenuse $D(t)$ and sides with length $D(t)$ and $W(t) - S(t)$, so the Pythagorean theorem states

$$D(t)^2 = 5280^2 + (W(t) - S(t))^2.$$

If we take the t derivative of both sides, we get

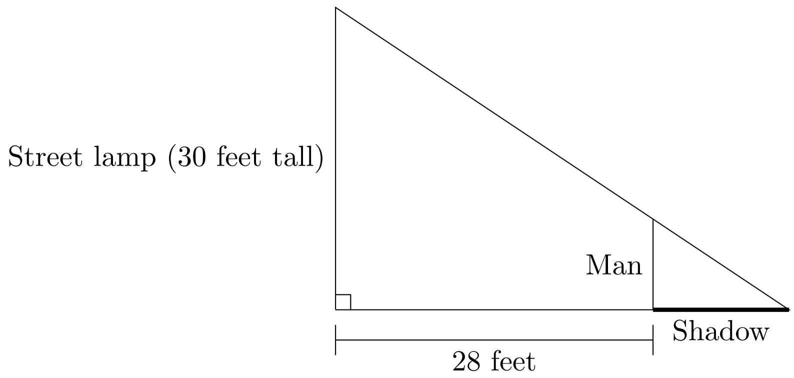
$$2D(t)\frac{dD}{dt} = 2(W(t) - S(t))\left(\frac{dW}{dt} - \frac{dS}{dt}\right).$$

To find $\left.\frac{dW}{dt}\right|_{t=15}$, we will need $D(15) = 5466.258$ and $\left.\frac{dD}{dt}\right|_{t=15} = 14.6468$. We also need to find $W(15) - S(15)$, but since this is one side of the right triangle, we can use tangent to find this distance: $W(15) - S(15) = 5280\tan(\pi/12) \approx 1414.7717$. Finally, we also need to know $\frac{dS}{dt}$, but in the description of the problem it says that ship is traveling at a constant speed of 6 ft/sec. Plugging all of this information into our equation we have

$$2(5466.258)(14.6468) = 2(1414.7717)\left(\frac{dW}{dt} - 6\right) \Rightarrow 56.5909 = \left.\frac{dW}{dt}\right|_{t=15} - 6.$$

Therefore, $\left.\frac{dW}{dt}\right|_{t=15} \approx 62.5909$ ft/sec

3. [8 points] A man, who is 28 feet away from a 30 foot tall street lamp, is sinking into quicksand. (*See diagram below.*) At the moment when 6 feet of him are above the ground, his height above the ground is shrinking at a rate of 2 feet/second.



Throughout this problem, remember to show your work clearly, and include units in your answers.

a. [3 points] How long will the man's shadow (shown in bold in the diagram above) be at the moment when 6 feet of him are above the ground?

Solution: Let s be the length of the shadow. Noticing that the larger and smaller triangles in the picture are similar triangles, we have

$$\begin{aligned}\frac{30}{28+s} &= \frac{6}{s}\\ 30s &= 168+6s\\ 24s &= 168\\ s &= 7.\end{aligned}$$

So the length of the shadow is 7 feet at that moment.

Answer: **7 feet**

b. [5 points] At what rate is the length of the man's shadow changing at the moment 6 feet of him are above the ground? Is his shadow growing or shrinking at that moment?

Solution: Let h be the height of the man above the ground, and let s be the length of his shadow. Using similar triangles as above, we have $\frac{30}{28+s} = \frac{h}{s}$ so $30s = 28h + hs$.

Taking derivatives with respect to time t, we find $30\frac{ds}{dt} = 28\frac{dh}{dt} + h\frac{ds}{dt} + s\frac{dh}{dt}$.

So at the moment when $h = 6$, we have

$$\begin{aligned}30\left.\frac{ds}{dt}\right|_{h=6} &= 28(-2) + 6\left.\frac{ds}{dt}\right|_{h=6} + 7(-2)\\ 24\left.\frac{ds}{dt}\right|_{h=6} &= -70\\ \left.\frac{ds}{dt}\right|_{h=6} &= \frac{-70}{24} = -\frac{35}{12} \approx -2.917\end{aligned}$$

So at that moment, the shadow is shrinking at a rate of about 2.917 feet/second.

Answer: The man's shadow is (circle one) GROWING [SHRINKING]

at a rate of $\frac{35}{12}$ **(about 2.917) feet/second** .