Math 115 — Practice for Exam 1

Generated September 4, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2006	1	9	Shakespeare	14	
Winter 2001	1	7	Michigan Arizona	16	
Fall 2001	1	7	cable	15	
Fall 2003	1	10	scale	12	
Fall 2004	1	9	arb	12	
Fall 2005	1	4	gasoline	12	
Total				81	

Recommended time (based on points): 73 minutes

9. Cliff is a stage manager for the Royal Shakespeare Company. He notices that the more people that are in the audience, the more nervous the actors get, and consequently they say their lines very fast and the play is shorter. Suppose $T(n)$ is a function that gives the running time of the play in minutes as a function of the number of people, n, in the audience. Cliff also remembers that the play ran 240 minutes at the last dress rehearsal when no one was in the audience and today they had a crowd of 300 people and the show ran for 2 hours.

   (a) (3 points) In the context of this problem what is the practical interpretation of $T(158)$?

   The expression $T(158)$ gives the number of minutes that the play will run when 158 people are in the audience.

   (b) (3 points) In the context of this problem what is the practical interpretation of $T'(158)$?

   The expression $T'(158)$ gives the approximate change in the number of minutes that the play will run when the audience increases from 158 to 159 people. The units of T' are minutes per person.

   (c) (4 points) Assuming that running time decreases linearly as the number of people attending increases, write an equation for the running time T in minutes as a function of n number of people attending.

   We are given two points, (0,240) and (300, 120). Thus, the slope of the linear function is

$$m = \frac{120 - 240}{300 - 0} = \frac{-120}{300} = -0.4.$$

   We are given the vertical intercept of (0,240), so an equation of the linear function is

$$T(n) = -0.4n + 240.$$

   (d) (4 points) Now assume instead that the running time is exponentially decreasing as a function of the number of the people in the audience, write an equation for the running time T in minutes as a function of the number of people n.

   In an equation of the form $T(n) = T_o a^n$, we are given that $T_o = 240$. Using the other given point, we have

$$120 = 240a^{300}$$

   so

$$a = 0.5^{1/300} = 0.9977$$

   Thus, if the running time is decreasing exponentially, an equation is

$$T(n) = 240(0.9977)^n.$$

7.) The populations of Michigan and Arizona between the years of 1960 and 1990 can be modeled by the following functions, where t is the number of years since 1960, and the units of the population is in millions.

Michigan: $f(t) = 7.8(1.0058)^t$; **Arizona:** $g(t) = 1.3(1.035)^t$

(a) (3 pts) [No sentence necessary.] Over the 30 year period, what was the annual percent growth rate for the population of Arizona?

3.5%

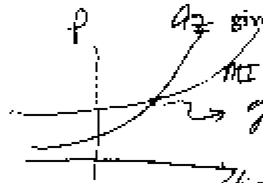
How much greater was that than the corresponding rate for Michigan?

$3.50 - 0.58 = 2.92\%$

(b) (2 pts) What was the difference in the two populations in 1960? [No sentence needed.]

$7.8 - 1.3 =$ 6.5 million people

(c) (4 pts) If the two states continue to grow according to the patterns given above, will there be a time when the population of Arizona will surpass that of Michigan? If not, explain (mathematically) why not. If so, give the year. [Show your work and express your answer in sentence form.]



$7.8(1.0058)^t = 1.3(1.035)^t \rightarrow \frac{7.8}{1.3} = \left(\frac{1.035}{1.0058}\right)^t \rightarrow \frac{\ln\left(\frac{7.8}{1.3}\right)}{\ln\left(\frac{1.035}{1.0058}\right)} = t \rightarrow t \approx 62.6$

$1960 + 62.6 = 2022.6$

Yes, in the year 2022 the population of Arizona would surpass the population of Michigan.

(d) (2 pts) How many people would the model predict for the population of Michigan in the 2000 census? [No sentence necessary—show work.]

In 2000, $t = 40$, so

$7.8(1.0058)^{40} =$ 9.83 million people

(e) (2 pts) Interpret, in the context of this problem, the *meaning* of $g^{-1}(2)$. [Sentence form, of course.]

In this model, $g^{-1}(2)$ gives the year that the population of Arizona will have 2 million people.

(f) (3 pts) According to the model above, in what year was the population of Michigan 5 million people? [Show work and express answer in sentence form.]

We want $5 = 7.8(1.0058)^t \rightarrow \frac{5}{7.8} = (1.0058)^t$, so $t \cdot \ln(1.0058) = \ln\left(\frac{5}{7.8}\right) \rightarrow t = \frac{\ln\left(\frac{5}{7.8}\right)}{\ln(1.0058)} \approx -76.89$

$1960.00 - 76.89 = 1883.11$

According to this model, the population of Michigan was 5 million people in the year 1883!

7. (15 pts) Your local cable internet provider has discovered that the strength (measured in Watts) of the signal generated at their broadcast station decreases fairly rapidly as it travels over their wires. They are concerned about this because subscribers must receive at least a 12-Watt signal in order for their systems to work. Engineers have calculated that $S = f(d) = 160(0.64)^d$, where S is the signal strength, and d represents distance (in miles) from the broadcast station.

a) Translate the statement "$f^{-1}(6) = 11$" into plain English. Is this statement true or false?

The distance from the source where the signal strength is 6 Watts is 11 miles. Not true if $f(d) = 160(0.64)^d$. For $d = 11$, $f(11)$ is much smaller than 6.

b) What percentage of the signal is lost over each mile of cable?

For each gain of 1 in d, the signal strength received is multiplied by 0.64. Thus, a fraction 0.36, or 36 %, is lost each mile.

c) What does the "160" tell you, in real world terms? (answer in a complete sentence.)

The initial signal is 160 Watts strong.

d) How far from the station can one live and still receive the service?

Since $f(d)$ is a decreasing function (exponential function with base $0.64 < 1$), we just solve for d in: $12 = f(d) = 160(0.64)^d$

But then $(0.64)^d = \frac{3}{40}$, $d = \ln(3/40) / \ln(0.64) = 5.804$

e) Better wiring is installed which cuts in half the percentage of the signal lost per mile. You find that the signal strength at your house is doubled! How far from the station do you live?

When this happens, the new function for S is

$g(d) = 160(0.82)^d$.

We want to solve $g(d) = 2f(d)$, or

$(0.82)^d = 2(0.64)^d$.

Take logs, and solve for d: $d = \dfrac{\ln 2}{\ln(0.82/0.64)} = 2.797.$

(10.) (12 points) When you weigh yourself by standing on a bathroom scale, you push down on a spring inside the scale. As the spring compresses – that is, as it decreases in length – your body is acted on by two forces:

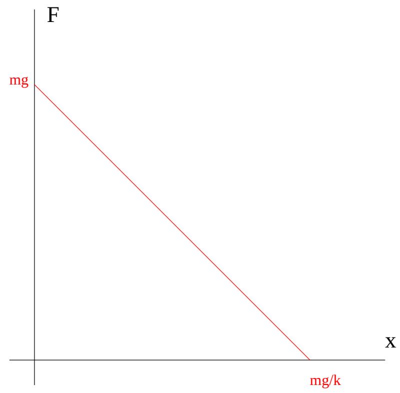
- Gravity exerts a downward force F_g on your body. The magnitude of this force is mg, where m is the mass of your body, and g is a constant.
- The spring in the scale exerts an upward force F_s on your body. The magnitude of this force is directly proportional to the total change in the spring's length. The constant of proportionality k is called the *spring constant* of the spring.

The net downward force F on your body equals the difference $F_g - F_s$.

(a) Write an expression for F as a function of x, the length by which the spring has been compressed.

$$F = mg - kx$$

(b) On the axes below, sketch a graph of F as a function of x, clearly labelling both intercepts.



(c) What is the significance of the x-intercept of this graph? Hint: we will refer to this x-value as x_{eq}.

When $x = x_{\text{eq}}$, the net force F is zero, because the gravitational force and the force from the spring are in *equilibrium*. Since there is no net force on your body, you stop sinking down into the scale.

(d) The mechanism inside the scale doesn't actually measure your mass m directly; instead, it measures the value of x_{eq}. However, it turns out that m and x_{eq} only differ by multiplication by a constant factor – that is, $m = c \cdot x_{\text{eq}}$, for some c. This means that the numbering of the scale's display can be chosen so that the scale gives a readout of your mass, after all.

What is the value of the constant c?

$$x_{\text{eq}} = \frac{mg}{k}, m = \left(\frac{k}{g}\right) x_{\text{eq}}$$

$$\boxed{c = \frac{k}{g}}$$

9. (12 points) As fall progresses the trees in the Arboretum gradually change color. The function f gives the percentage of leaves on a particular tree that have began to change colors as a function of the number of days since September 30. (October 1st corresponds to $t = 1$.) All answers should be in complete sentences.

(a) Give a practical interpretation in everyday terms describing what $f(10) = 15$ means in the context of this problem.

On October 10th 15% of the leaves have began to change color.

(b) Give a practical interpretation for what $f'(15) = 9$ means in the context of this problem.

On October 15th approximately 9% more of the tree's leaves will begin to change color during the day.

(c) Give a practical interpretation for what $f^{-1}(3) = 6$ means in the context of this problem.

When 3% of the leaves have began to change colors it is October 6th.

(d) Give a practical interpretation describing what $(f^{-1})'(40) = 0.5$ means in the context of this problem.

When 40% of the tree's leaves have begun to change colors, it will take approximately 12 hours for the next percent to change colors.

4. (12 points) The cost of gasoline has risen dramatically in the last six months. At the beginning of March, the cost of gasoline was \$2.10 per gallon, but at the beginning of September, the cost was \$3.00 per gallon.

(a) Suppose the cost of gasoline, C, measured in dollars per gallon is a linear function of time, t. Find a formula for cost of gasoline as a function t, in months, since the beginning of March.

Since $t = 0$ represents March, we have the vertical intercept of 2.10. The slope can be found by

$$\frac{\Delta C}{\Delta t} = \frac{3.00 - 2.10}{6} = \frac{.90}{6} = 0.15$$

Thus, $C(t) = 0.15t + 2.10$.

(b) Suppose further that you drive 200 miles per month and that your car averages 27 miles per gallon. Use your formula from part (a) to calculate the price of gasoline at the beginning of December. Assuming that the cost stays the same throughout the month of December, calculate your gas cost for the month of December.

In December, $t = 9$, so using our formula from (a), the cost of gas at the beginning of December will be $C(9) = \$3.45$ per gallon. We will use $\frac{200}{27} = 7.41$ gallons. Therefore, our total cost will be

$$7.41 \times \$3.45 = \$25.56$$

(c) Now suppose instead that C is an exponential function of time. Find a formula for the cost of gas as a function of t, in months since March.

The form of our function is:

$$C(t) = C_0 B^t$$

with $C_0 = 2.10$. Using the point $(6, 3.00)$, we have

$$3 = 2.10(B)^6$$

which gives $B = 1.0612$. Thus, if the increase is exponential, $C(t) = 2.10(1.0612)^t$.

(d) Use your formula from part (c) to calculate the cost of gasoline at the beginning of December. Assuming that this cost remains the same throughout December, use the milage information from part (b) to calculate your total gasoline cost for December.

Using the formula from (c) we find that the cost of gasoline at the beginning of December is $C(9) = \$3.58$ per gallon. Therefore, the total cost is

$$7.41 \times \$3.58 = \$26.56$$