Math 115 — Practice for Exam 1

Generated September 29, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2014	1	7	Waterville	15	
Fall 2016	1	5	deep temp	10	
Winter 2016	1	5	train	10	
Fall 2011	1	4	Twitter	12	
Winter 2011	1	8	solar array	12	
Winter 2012	1	8	advertising	9	
Total				68	

Recommended time (based on points): 61 minutes

7. [15 points] During the winter, the town of Waterville uses salt to keep the roads from freezing. Let $S = f(T)$ be the amount of salt, in tons, used on the roads of Waterville on a day when the average temperature is T °F. Let $C = g(S)$ be the cost, in thousands of dollars, of S tons of salt. Assume that both f and g are invertible functions that are differentiable everywhere.

a. [3 points] Interpret the equation $f^{-1}(4) = 9$ in the context of this problem.
Use a complete sentence and include units.

Solution: On a day when Waterville uses 4 tons of salt on the roads, the average temperature is 9 °F.

b. [3 points] Interpret the equation $g(f(7)) = 2$ in the context of this problem.
Use a complete sentence and include units.

Solution: On a day when the average temperature is 7 °F, the salt used on Waterville's roads costs $2000.

c. [2 points] Yesterday, the average temperature in Waterville was w °F.
Give a single mathematical expression equal to the average temperature, in °F, on a day when Waterville uses twice as much salt on the roads as it did yesterday.

Answer: $f^{-1}(2f(w))$

d. [4 points] Give a single mathematical equality involving the derivative of f which supports the following claim:
On a day when the average temperature is 3°F, Waterville uses approximately 0.12 tons less salt on the roads than on a day when the average temperature is 1°F.

Answer: $f'(1) = -0.06$

e. [3 points] In the equation $(g^{-1})'(8) = 5$, what are the units on 8 and 5?

Answer: Units on 8 are **thousands of dollars**

Answer: Units on 5 are **tons of salt per thousand dollars**

5. [10 points] Scientists bore a hole deep into the earth and lower an instrument to record the temperature. As the instrument goes deeper, the temperature it records increases. Let $T = g(w)$ be the temperature, in degrees Celsius, the instrument records when it is w hectometers below the surface of the earth. (Recall that 1 hectometer is 100 meters.) Assume that the function g is invertible and that the functions g and g^{-1} are continuous and differentiable.

a. [3 points] Using a complete sentence, give a practical interpretation of the equation $g^{-1}(68) = 49$ in the context of this problem. Be sure to include units.

Solution: When the instrument records a temperature of 68°C, the instrument is 4.9 kilometers below the surface of the earth.

b. [4 points] Below is the first part of a sentence that will give a practical interpretation of the equation $g'(13) = 0.6$ in the context of this problem. Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include units in your answer.
When the instrument is lowered from 1300 meters to 1320 meters below the surface of the earth, the temperature it records . . .

Solution: increases by approximately 0.12 Celsius degrees.

c. [3 points] Circle the one statement below that is best supported by the equation

$$(g^{-1})'(56) = 0.4.$$

i. The temperature recorded by the instrument is 56°C when it is about 0.4 hectometers below the surface of the earth.

ii. The temperature recorded by the instrument increases from 56°C to 56.4°C when the instrument is lowered approximately one more hectometer.

iii. When the instrument is lowered from 55.9 hectometers to 56 hectometers below the surface of the earth, it detects an increase in temperature of about 0.04 Celsius degrees.

iv. The temperature recorded by the instrument increases from 56°C and 57°C when the instrument is lowered about $\frac{1}{0.4}$ (= 2.5) hectometers further.

v. As the temperature recorded by the instrument increases from 55.9°C to 56°C, the instrument is lowered about 4 meters further beneath the surface of the earth.

vi. When the instrument is 56 hectometers below the surface of the earth, the recorded temperature is increasing at a rate of 0.04 Celsius degrees per meter.

5. [10 points] Vikram takes a non-stop train ride from Chennai straight to New Delhi. Let $g(t)$ be the distance (in km) of Vikram's train from Chennai t hours after his ride begins. Assume that the function g is increasing and invertible, and that g and g^{-1} are differentiable. Several values for $g(t)$ are shown in the table below.

t	0	2	5	6.5	10	11	16	28
$g(t)$	0	132	346	448	692	742	1152	2180

a. [3 points] Estimate the instantaneous velocity of Vikram's train 6 hours after his ride begins. *Show your work and include units.*

Solution: We estimate using average velocity based on nearby measurements:

$$\frac{448-346}{6.5-5} = 68$$

So we estimate the instantaneous velocity of Vikram's train 6 hours after his ride begins to be about 68 km/h.

Answer: 68 **km/h**

b. [5 points] Suppose $(g^{-1})'(700) = C$, where C is some constant.

(i) Using the data in the table above, find the best possible estimate of C. *Show your work.*

Solution: We estimate the derivative based on nearby measurements:

$$\frac{11-10}{742-692} = 0.02 \text{ h/km}$$

So we estimate $C \approx 0.02$. 700 has units km, and C has units h/km.

Answer: 0.02

(ii) In interpreting the equation $(g^{-1})'(700) = C$, what are the units on 700 and C?

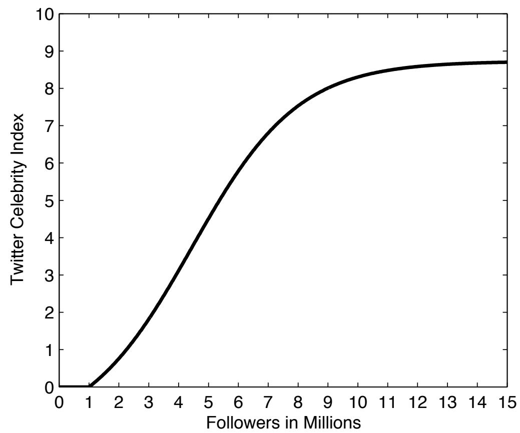
Answer: Units on 700 are **km**

Answer: Units on C are **h/km**

c. [2 points] Let $R(t)$ be the total rainfall (in cm) in New Delhi during the first t hours of Vikram's train ride. Express the following statement with a single mathematical equation: "Over the first 900 km of Vikram's train ride, it rained 3.6 cm in New Delhi."

Answer: $R(g^{-1}(900)) = 3.6$

4. [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.



Use the graph above to help you answer the following questions.

a. [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

Solution: When a Twitter user has 13.72 million followers, their Twitter Celebrity index is 8.67.

b. [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

Solution: When a user has a Twitter Celebrity index of 4.25, they have 4.88 million followers.

c. [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

Solution: When a Twitter user has 10 million followers, adding $100,000$ followers will increase their celebrity index by roughly .02278.

d. [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

Solution: When a Twitter user has celebrity index 7.238, and increase of .1 to their index corresponds to gaining approximately .071 million (71,000) followers.

8. [12 points] Let $P(d)$ be a function giving the total electricity that a solar array has generated, in kWH, between the start of the year and the end of the dth day of the year. Each of the following centences (a)–(d) expresses a mathematical equality in practical terms. For each, give a **single** mathematical equality involving P (and, as needed, its inverse and derivatives) that corresponds to the sentence.

a. [3 points] The end of the day on which the array had generated 3500 kWH of electricity was the end of the 4th of January.

Solution: $P^{-1}(3500) = 4$. ($P(4) = 3500$ is equivalent, though the statement suggests that 3500 kWH is the independent variable, and so that the equation involving the inverse is the better for this statement.)

b. [3 points] At the end of January 4th, the array was generating electricity at a rate of 1000 kWH per day.

Solution: $P'(4) = 1000$.

c. [3 points] When the array had generated 5000 kWH of electricity, it would take approximately half a day to generate an additional 1000 kWH of electricity.

Solution: $(P^{-1})'(5000) = \frac{1}{2000}$. (Or, alternately, $P'(P^{-1}(5000)) = 2000$.)

d. [3 points] At the end of January 30th, it would take approximately one day to generate an additional 2500 kWH of electricity.

Solution: $P'(30) = 2500$. (Or, $P'(31) = 2500$.)

8. [9 points] A certain company's revenue R (in thousands of dollars) is given as a function of the amount of money a (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that f is invertible.

a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

- If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
- If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
- If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
- If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

Solution: The last option.

b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

Solution:
They should spend less on advertising, because if they increase their advertising expenditure by \$1000, they will only gain about \$500 in revenue.

c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a\ln(a+1).$$

Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

Solution:

$$f'(100) = \lim_{h\to 0} \frac{(100+h)\ln(100+h+1) - 100\ln(101)}{h}$$